Takagi-Sugeno model-based design for switching systems and local stabilization

Habilitation thesis

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Part I

Preliminaries

Chapter 1

Introduction

1.1 Research overview

My research work concerns developing automated methods for stability analysis, controller and observer design for nonlinear systems represented by Takagi-Sugeno (TS) (Takagi and Sugeno, 1985) fuzzy models. The TS model is a convex combination of linear models. This structure facilitates analysis and design by using effective algorithms based on Lyapunov functions and linear matrix inequalities. TS models are known to be universal approximators and, in addition, a broad class of nonlinear systems can be exactly represented as a TS system. In the last three decades, within a mathematically rigorous framework, several results concerning such models have been developed. The TS representation in principle can adapt the advantages of linear time-invariant system-based design to nonlinear systems.

Analysis and controller and observer synthesis for TS models are usually performed using Lyapunov's direct method, employing – classically – a quadratic, more recently, nonquadratic Lyapunov functions. Since there are efficient algorithms to solve linear matrix inequalities (LMIs), the goal is to develop the stability or design conditions in such a form. My recent works involve analysis and design for discrete-time TS systems.

In the discrete time case, non-quadratic Lyapunov functions brought a significant improvement (Guerra and Vermeiren, 2004; Ding et al., 2006; Dong and Yang, 2009; Lee et al., 2011) in the development of global stability and design conditions. They also allow some relaxations by calculating this difference between α – instead of the usual two – consecutive instants (Kruszewski et al., 2008) or by considering delayed Lyapunov functions (Lendek et al., 2015). It has been proven that the solutions obtained by non-quadratic Lyapunov functions include and extend the set of solutions obtained using the quadratic framework. More recently, by using Polya's theorem (Montagner et al., 2007; Sala and Ariño, 2007) asymptotically necessary and sufficient (ANS) LMI conditions have been obtained for stability in the sense of a chosen quadratic or nonquadratic Lyapunov function. Ding (2010) gave ANS stability conditions for general membership function-dependent Lyapunov matrix. By increasing the complexity of the homogeneously polynomially parameter-dependent Lyapunov functions, in theory any sufficiently smooth Lyapunov function can be approximated. Unfortunately, the number of LMIs that have to be solved increase quickly, leading to numerical intractability (Zou and Yu, 2014). For all these results, if the developed conditions are feasible, then stability of the corresponding system is ensured globally – actually in the largest Lyapunov level set included in the domain where the TS model is defined. However, it is possible that stability (of the closed-loop system or error dynamics, as might be the case) cannot be ensured on the full domain where the TS model is defined. Thus, the control and estimation problems remain highly challenging due to the computational limitations and conservatism of existing methods.

My postdoctoral and more recent work focuses strongly on design methods for systems exhibiting a structure, such as periodic or switching. My main goal was to incorporate the known structure in the design steps and in this way reduce the conservativeness of the conditions for computing the controller.

Periodic and switching models can be found in numerous domains such as automotive, aeronautic, aerospace and even computer control of industrial process. For example in (Chauvin et al., 2005), a periodic dynamic model is used to estimate the air/fuel ratio in each cylinder on an internal combustion engine, Gaiani et al. (2004) proposed a periodic model for the rotor blades of helicopter, etc. Switching models are also frequently encountered in control problems concerning HCCI combustion (Liao et al., 2013), air path with turbocharger (Nguyen et al., 2012b,a, 2013), clutch actuator control (Langjord et al., 2008).

In the literature, mainly continuous time switching systems were considered but recently discrete time approaches have also been developed (Chen et al., 2012; Duan and Wu, 2012; Hetel et al., 2011). Stabilization and tracking conditions for continuous-time linear switching systems have been developed in (Baglietto et al., 2013; Battistelli, 2013), delay-dependent stabilization in (Kim et al., 2008), and observability with unknown input has been investigated in (Boukhobza and Hamelin, 2011).

The first major part of the thesis deals with discrete-time nonlinear periodic and switching systems represented by TS models, where the subsystems may be unstable or uncontrollable/ unobservable. Using a periodic or switching non-quadratic Lyapunov function, conditions to establish stability and to design observers and controller are presented. Furthermore, it is well-known that by switching between two – independently stable – subsystems the switching system can be destabilized and conversely, by switching between unstable subsystems, the states can be made to converge to zero. The design of a switching law that stabilizes a given switching system – and the conditions under which such a law exists – is also presented.

The second major part of this thesis deals with a different problem. Many nonlinear systems have several equilibrium points, thus existing methods in the discrete-time TS framework – which involve global conditions – cannot establish stability of such points. Furthermore, one of the main points of interest is the determination of an estimate of the domain of attraction of an equilibrium point. Using a quadratic Lyapunov function, a quadratic domain may be obtained, but this can be restrictive compared to the true domain of attraction. Current nonquadratic results, such as the one in (Lee et al., 2013) requires the knowledge of some a priori bounds on the variation of the membership functions over time, while in (Lee and Joo, 2014) the bounds on the derivative of the membership function with respect to the scheduling variables are introduced in the stability conditions. This procedure is complex and can be computationally expensive, especially for practical implementation. Thus, this part of the thesis presents methods for local stability analysis and local controller and observer design for TS models. The tools existing in the discrete TS framework are combined with the determination of a non-quadratic domain of attraction using an easy procedure requiring only the knowledge of the membership functions.

The above two directions, switching systems and local analysis and design, comprise the main thrust of my post-PhD research. Additional research directions, and applications are reviewed in separate chapters. Direct offshoots of my PhD research are not discussed, even if they were done or published after the PhD. The same applies to work where I participated but did not take a leading role.

1.2 Advising and management activities

Throughout my research I have been managing several student projects, in the Netherlands, in France and in Romania. I have been leading as coadvisor 2 PhD students, one in the Netherlands and one in France, one of whom graduated in 2013 and one in 2015; I have also co-advised 2 visiting PhD students at TUDelft and 1 MSc student; and 4 Master and 15 Bachelor students in Romania. With my students, I investigated an agenda of control topics and applications complementary to my main research lines, see Chapter 12.

I have successfully acquired funding for my research in three national projects funding young teams (projects PN-II-RU-TE-2011-3-0043, PN-II-RU-TE-2014-3-0942, and PN-III-P1-1.1-TE-2016-1265) over the period 2011-2018; one international cooperation project (ESA 4000123993/18/NL/CRS); as well as an internal grant within a 2013 funding initiative at the Technical University of Cluj-Napoca, Romania. I have also been involved in local project management in French, Dutch, and Romanian research projects.

I have organized and co-organized several successful special sessions at the IEEE International Conference on Fuzzy Systems, IEEE World Congress on Computational Intelligence and IFAC World Congress; I am Associate Editor at Engineering Applications of Artificial Intelligence (IF 2.819), and have been Editor at the 4th IFAC International Conference on Intelligent Control and Automation Sciences, 2016; special session chair at the IEEE International Conference on Automation, Quality and Testing, Robotics, 2014, 2016, etc.

1.3 Teaching activities

In my work so far, I have been involved in the Dutch and Romanian academic systems. My teaching career has begun in 2004, during my postgraduate studies at the Technical University of Cluj-Napoca, when I taught practical classes for the disciplines *Optimization techniques*, and *Control theory*. At the same place, but in the complementary role of student, I was exposed directly to the scientific literature, in course projects that required the critical investigation and evaluation of methods proposed in published papers.

Later, at the Technical University of Delft, as a course assistant at DCSC, I have been teaching practical sessions and developing assignments for the course *Modeling and system analysis*. For the relatively small groups (up to 50 students), constant interaction, entailing open questions, searching for possible applications, exercise solving, relating to state-of-the-art results in the literature, bonus questions were a way both for motivating the students and their efficient learning incited researching and innovation.

My *independent* teaching career started in 2011 at the Technical University of Cluj-Napoca, where I was first lecturer (October 2011), then associate professor (October 2013). I am currently leading the disciplines *Optimization* and *Estimation for control*, including lecture, laboratory, project, and examination work.

I have also been invited to present my research at universities from France and Spain. Overall, my expertise familiarized me with the full spectrum of teaching skills needed in a University teaching career.

1.4 Outline of the thesis

This thesis is structured as follows. After the present introduction, Part I describes some necessary background, and the use of general delayed Lyapunov functions for control. The main content is structured in two parts along the two major threads of work presented above. Namely, Part II focuses on developments in analysis and design for switching systems, while Part III on local analysis and design. At the end, in Part IV, other research directions are outlined, and an overall plan for the future is delineated. For easy reference, local lists of references are provided at the end of each part.

Both of the main Parts, II and III, are structured in a similar way: after a brief outline, the major research contributions in that direction are presented, in separate chapters for stability analysis, controller design, and observer design. The results in each case are discussed and illustrated on examples in the particular chapter; extensions and generalizations are also presented in the corresponding chapters.

The two parts can be read independently. Since in effect the controller and observer design in each part are extensions from stability analysis, the corresponding analysis chapter is needed to understand them. Furthermore, the generalizations make use of the results presented in Chapter 3. Within each chapter, some occasional backtracking may be necessary to find the description of examples used multiple times, which are only presented in detail once, the first time they are used.

1.5 Acknowledgments

I am grateful to all my students and collaborators for their contributions to the work presented here. Among the students, I would like to mention especially my PhD students Victor Estrada-Manzo and Pawel Stano; and my Master students Zoltán Nagy and Paul Petrehuş, all of whom did significant work ending up in journal and conference publications. Long-term, extremely fruitful collaborations with Thierry-Marie Guerra and Jimmy Lauber should also be highlighted; and of course, the contribution of my PhD advisors Robert Babuška and Bart De Schutter with whom we are continuing to work together and coauthor papers. Countless other people have contributed, formally or informally, with scientific ideas, material, and opportunities. And, naturally, without the help of family and friends this thesis would not exist. Thank you all!

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Finally, much of this thesis is based on existing papers and books with several publishers, including among others IEEE, Elsevier, Springer, and Wiley. The copyright for the material remains with the respective publishers, and I am grateful to them for hosting my publications and reusing the material here.

To recognize that the work presented here is the result of the concerted effort and contributions of all these individuals and organizations, the remainder of the thesis will be written in the first person plural.

Chapter 2

Background

Several technical preliminaries from the preexisting literature are necessary to explain the results presented in this thesis. We start by presenting the Takagi-Sugeno (TS) fuzzy models and the notations we use throughout the thesis and the properties and lemmas we employ. This chapter also presents the basic results from the literature regarding stability analysis of discrete-time TS models and the way the conditions are obtained.

2.1 Takagi-Sugeno models

Since the introduction of the concept of fuzzy sets by Zadeh (1965), fuzzy sets, fuzzy logic, and more recently, fuzzy systems have found their use in a large set of applications (Klir and Yuan, 1995). They have been used extensively in expert systems, applications that required a linguistic description.

A different field in which fuzzy systems have been extensively employed and brought significant improvements is nonlinear control (Tanaka and Wang, 2001). In this case, the usual "fuzzy", linguistic interpretation no longer holds, but the name fuzzy is used to denote the combination of systems. The domain has matured in the last two decades and now an efficient and useful framework exists.

In the research described hereafter, we consider the use of fuzzy systems, specifically dynamic Takagi-Sugeno fuzzy models, for the analysis and design of discrete-time nonlinear systems of the form

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{f}(\boldsymbol{x}(k), \boldsymbol{u}(k), \boldsymbol{\theta}(k)) \\ \boldsymbol{y}(k) &= \boldsymbol{g}(\boldsymbol{x}(k), \boldsymbol{u}(k), \boldsymbol{\zeta}(k)) \end{aligned} \tag{2.1}$$

where $\boldsymbol{f}$ denotes the state transition function, describing the evolution of the states over time, $\boldsymbol{g}$ is the measurement function, relating the measurements to the states, $\boldsymbol{x}$ is the vector of the state variables, $\boldsymbol{u}$ is the vector of the input or control variables, $\boldsymbol{y}$

denotes the measurement vector, and $\boldsymbol{\theta}$ and $\boldsymbol{\zeta}$ are unknown or uncertain parameters or disturbances.

We represent the system above by Takagi-Sugeno (TS) fuzzy model of the form

$$\begin{aligned} \boldsymbol{x}(k+1) &= \sum_{i=1}^{r} h_i(\boldsymbol{z}(k))(A_i\boldsymbol{x}(k) + B_i\boldsymbol{u}(k) + a_i) \\ \boldsymbol{y}(k) &= \sum_{i=1}^{r} h_i(\boldsymbol{z}(k))(C_i\boldsymbol{x}(k) + c_i) \end{aligned} \tag{2.2}$$

where r is the number of local models, A_i, B_i, C_i, are the matrices and a_i and c_i are the biases of the ith local model, $\boldsymbol{z}$ is the vector of the scheduling variables, which may depend on the states, inputs, measurements, or other exogenous variables, and h_i, $i = 1, 2, \ldots, r$ are normalized membership functions, i.e., $h_i(\boldsymbol{z}) \geq 0$ and $\sum_{i=1}^{r} h_i(\boldsymbol{z}) = 1$, $\forall k \in \mathbb{N}$.

Such a model presents several advantages. The TS model is a universal approximator (Fantuzzi and Rovatti, 1996), and many nonlinear systems can be exactly represented in a compact set of state variables as TS systems (Ohtake et al., 2001). Moreover, (2.2) is the convex combination of local affine models, which facilitates stability analysis and controller and observer design for such systems. In addition, many stability and design conditions for TS systems can be formulated as linear matrix inequalities (Boyd et al., 1994; Scherer and Weiland, 2005; Tanaka and Wang, 1997; Tanaka et al., 1998), for which efficient algorithms exist.

Two main approaches can be used to obtain TS fuzzy models: 1) identifying the model using measured or simulated data and 2) analytic construction of a TS model that exactly represents or approximates a given nonlinear dynamic model.

Here we consider the second approach, i.e., constructing a TS model from the nonlinear dynamic model. For methods for identification of TS systems the interested reader is referred to (Driankov et al., 1993; Abonyi et al., 2002; Babuška et al., 2002; Johansen and Babuška, 2003; Kukolj and Levi, 2004; Kaymak and van den Berg, 2004; Angelov and Filev, 2004).

Several methods exist that construct a fuzzy representation or an approximation of a given model. Using the method described in Chapter 14 of (Tanaka and Wang, 2001) a TS fuzzy model can be constructed such that both the nonlinear system and its derivative are approximated. Other methods that approximate a given nonlinear system are dynamic linearization (Johansen et al., 2000), which is in fact a Taylor series expansion in several operating points, or the substitution method developed by Kiriakidis (2007). Furthermore, when variables are defined on a compact set, TS models have been proven to be able to approximate any nonlinear function to an arbitrary degree of accuracy (Wang and Mendel, 1992; Kosko, 1994; Ying, 1994; Fantuzzi and Rovatti, 1996).

The sector nonlinearity approach (Ohtake et al., 2001) can be employed to obtain a TS model that is an exact representation of a given nonlinear system in a considered compact set. Note, however, that this representation is not unique. Furthermore, depending on the choices made, the number of local models may be exponential in the number of nonlinearities, which may lead to intractable design problems. Therefore, unless instability, unobservability, or uncontrollability of the local models is an issue, a representation with a minimum number of rules should be chosen. Moreover, the obtained consequent models are not guaranteed to be stable, observable, or controllable. Classic methods to analyze stability of a TS model require that the linear local models are stable. Likewise, the methods for controller/observer design require that the local models are controllable/observable. Depending on the nonlinear system considered and the goal – analysis or design –, another representation or an approximation of the nonlinear system may need to be chosen to have the desired properties.

Once a TS representation or approximation of the nonlinear system (2.1) is available, the analysis is performed using Lyapunov's direct method. At first, common quadratic Lyapunov functions have been used and conditions have been developed independently of the membership functions. In the last years, results in the discrete-time case have been significantly improved by the use of nonquadratic Lyapunov functions. With their use more evolved state-feedback controllers and observers were developed, and the design conditions became less conservative.

Although results exist, in the literature, affine local models are rarely used for controller design, since, using current control design methods, the affine terms have to be compensated for in each rule, which is possible only in special cases. For observer design, affine local models do not present a problem. However, to keep he same type of models, in what follows, TS fuzzy models with linear consequents will be used, i.e., instead of (2.2) we have the form

$$\begin{aligned} \boldsymbol{x}(k+1) &= \sum_{i=1}^{r} h_i(\boldsymbol{z}(k))(A_i\boldsymbol{x}(k) + B_i\boldsymbol{u}(k)) \\ \boldsymbol{y}(k) &= \sum_{i=1}^{r} h_i(\boldsymbol{z}(k))(C_i\boldsymbol{x}(k)) \end{aligned} \tag{2.3}$$

In this case all the local models have the same equilibrium point, zero, and such, Lyapunov stability analysis can naturally be employed. We also assume that the membership functions – in case of controller design – do not depend on the control input, in order to avoid having to solve implicit equations. For observer design, the input is considered as a known (measured) variable, and therefore the membership functions may depend on it.

2.2 Notations

For analysis as well as design the conditions are usually formulated as Linear Matrix Inequalities (LMIs), for which efficient algorithms exist. We present a brief overview of their useful properties.

Let $F = F^T \in \mathbb{R}^{n\times n}$ be a symmetric matrix. In what follows, we use $F > 0$ and $F < 0$, respectively, for positive and negative definiteness, meaning that every eigenvalue of F is strictly positive (negative). In general, whenever an expression is written as $F > 0$, it is assumed that the expression is symmetric, i.e., $F = F^T > 0$, even if the explicit symmetry condition is omitted.

Let $A, B \in \mathbb{R}^{n\times n}$ be two symmetric matrices. Then $A > B$ is equivalent to $A - B > 0$.

A star $(*)$ in a matrix indicates a transposed quantity in the symmetric position. For instance, $\begin{pmatrix} P & (*) \\ A & \tilde{P} \end{pmatrix} < 0$ is equivalent to $\begin{pmatrix} P & A^T \\ A & \tilde{P} \end{pmatrix} < 0$, $A^T P(*)$ is equivalent to $A^T PA$ and $A + (*)$ is equivalent to $A + A^T$. 0 and I denote the zero and identity matrices of appropriate dimensions.

In what follows we also use the notation X_z for the single sum $X_z = \sum_{i=1}^r h_i(\boldsymbol{z}(k))X_i$, X_{zz} to denote the double sum $X_{zz} = \sum_{i=1}^r \sum_{j=1}^r h_i(\boldsymbol{z}(k))h_j(\boldsymbol{z}(k))X_{ij}$, X_z^{-1} for the inverse of the sum $X_z^{-1} = (\sum_{i=1}^r h_i(\boldsymbol{z}(k))X_i)^{-1}$, etc., where h stands for the membership functions and $\boldsymbol{z}(k)$ for the scheduling vector at sample k. We also use the notations $X_{z+} = \sum_{i=1}^r h_i(\boldsymbol{z}(k+1))X_i$, $X_{z-} = \sum_{i=1}^r h_i(\boldsymbol{z}(k-1))X_i$, and $X_{z+\alpha} = \sum_{i=1}^r h_i(\boldsymbol{z}(k+\alpha))X_i$, $\alpha \in \mathcal{N}$, when the scheduling variables are evaluated in other sampling instants.

For the general case of multiple sums we use the following notations.

Definition 2.1 *(Multiple sum) We denote a multiple sum with n_P terms and delays evaluated at time k of the form*

$$\mathbb{P}_{G_0^P} = \sum_{i_1=1}^r h_{i_1}(\boldsymbol{z}(k+d_1)) \sum_{i_2=1}^r h_{i_2}(\boldsymbol{z}(k+d_2)) \sum_{i_{n_P}=1}^r h_{i_{n_P}}(\boldsymbol{z}(k+d_{n_P}))P_{i_1 i_2 \dots i_{n_P}}$$

where G_0^P is the multiset of delays $G_0^P = \{d_1, d_2, \dots, d_{n_P}\}$.

Definition 2.2 *(Multiset of delays) G_0^P denotes the multiset containing the delays in the multiple sum involving P at time k. G_α^P denotes the multiset containing the delays in the sum P at time $k+\alpha$.*

Note that we use ordered multisets, i.e., $d_{i+1} \leq d_i$, $\forall i \in \{1, 2, \dots, n_P\}$.

Definition 2.3 *(Cardinality) The cardinality of a multiset G, $|G|$, is defined as the number of elements in G.*

Definition 2.4 *(Index set) The index set of a multiple sum $\mathbb{P}_G$ is $\mathbb{I}_G = \{i_j | i_j = 1, 2, \dots, r,\ j = 1, 2, \dots, |G|\}$, the set of all indices that appear in the sum. Note that these indices are directly related to the delays in G. An element $\boldsymbol{i} \in \mathbb{I}_G$ is a multiindex.*

Definition 2.5 *(Multiplicity) The multiplicity of an element x in a multiset G, $\mathbf{1}_G(x)$ denotes the number of times this element appears in the multiset G.*

Definition 2.6 *(Union) The union of two multisets G_A and G_B is $G_C = G_A \cup G_B$ such that $\forall x \in G_C$, $\mathbf{1}_{G_C}(x) = \max\{\mathbf{1}_{G_A}(x), \mathbf{1}_{G_B}(x)\}$.*

Definition 2.7 *(Intersection) The intersection of two multisets G_A and G_B is $G_C = G_A \cap G_B$ such that $\forall x \in G_C$, $\mathbf{1}_{G_C}(x) = \min\{\mathbf{1}_{G_A}(x), \mathbf{1}_{G_B}(x)\}$.*

Definition 2.8 *(Sum) The sum of two multisets G_A and G_B is $G_C = G_A \oplus G_B$ such that $\forall x \in G_C$, $\mathbf{1}_{G_C}(x) = \mathbf{1}_{G_A}(x) + \mathbf{1}_{G_B}(x)$.*

Definition 2.9 *(Projection of an index) The projection of the index $\boldsymbol{i} \in \mathbb{I}_{G_A}$, to the multiset of delays G_B, $pr^{\boldsymbol{i}}_{G_B}$ is the part of the index that corresponds to the delays in $G_A \cap G_B$.*

Example 2.1 Consider the multiple sum

$$\sum_{i_1=1}^{r} h_{i_1}(\boldsymbol{z}(k)) \sum_{i_2=1}^{r} h_{i_2}(\boldsymbol{z}(k)) \sum_{i_3=1}^{r} h_{i_3}(\boldsymbol{z}(k-1)) \sum_{i_4=1}^{r} h_{i_4}(\boldsymbol{z}(k-2)) P_{i_1 i_2 i_3 i_4}$$

G_0^P is given by $G_0^P = \{0, 0, -1, -2\}$, $G_\alpha^P = \{\alpha, \alpha, \alpha-1, \alpha-2\}$. The index set of the multiple sum $\mathbb{P}_{G_0^P}$ is $\mathbb{I}_{G_0^P} = \{i_j | i_j = 1, 2, \ldots, r,\ j = 1, 2, 3, 4\}$.

The cardinality of G_0^P is $|G_0^P| = 4$. The multiplicity of the elements are $\mathbf{1}_{G_0^P}(0) = 2$, $\mathbf{1}_{G_0^P}(-1) = 1$, $\mathbf{1}_{G_0^P}(1) = 0$.

Consider now two multisets $G_A = \{0, 0, -1, -2\}$, and $G_B = \{0, 0, -1, -1\}$. The union of these two multisets is $G_A \cup G_B = \{0, 0, -1, -1, -2\}$, the intersection is $G_A \cap G_B = \{0, 0, -1\}$, and their sum is $G_A \oplus G_B = \{0, 0, 0, 0-1, -1, -1, -2\}$.

Note that the projection of a multiindex is in general not unique. For instance, the projection of an element $\boldsymbol{i} \in \mathbb{I}_{G_0^P}$ to $G_C = \{0, -1\}$ is either $\mathrm{pr}^{\boldsymbol{i}}_{G_C} = \{i_1, i_3 | i_1, i_3 = 1, 2, \ldots, r\}$ or $\mathrm{pr}^{\boldsymbol{i}}_{G_C} = \{i_2, i_3 | i_2, i_3 = 1, 2, \ldots, r\}$. □

2.3 Useful properties and lemmas

In what follows, let us recall some useful properties that will be used further on.

Property 1 (Congruence) Given a matrix $P = P^T$ and a full column rank matrix Q it holds that

$$P > 0 \;\Rightarrow\; QPQ^T > 0$$

Property 2 (Schur complement) Consider a matrix $M = M^T = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{pmatrix}$, with M_{11} and M_{22} being square matrices. Then

$$M < 0 \Leftrightarrow \begin{cases} M_{11} < 0 \\ M_{22} - M_{12}^T M_{11}^{-1} M_{12} < 0 \end{cases} \Leftrightarrow \begin{cases} M_{22} < 0 \\ M_{11} - M_{12} M_{22}^{-1} M_{12}^T < 0 \end{cases}$$

Property 3 (S-procedure) Consider matrices $F_i = F_i^T \in \mathbb{R}^{n \times n}$, $\boldsymbol{x} \in \mathbb{R}^n$, such that $\boldsymbol{x}^T F_i \boldsymbol{x} \geq 0$, $\forall \boldsymbol{x}$, $i = 1, \ldots, p$, and the quadratic inequality condition

$$\boldsymbol{x}^T F_0 \boldsymbol{x} > 0 \tag{2.4}$$

$\boldsymbol{x} \neq 0$. A sufficient condition for (2.4) to hold is: there exist $\tau_i \geq 0$, $i = 1, \ldots, p$, such that $F_0 - \sum_{i=1}^{p} \tau_i F_i > 0$.

Property 4 (Completion of squares) Given two matrices X and Y of proper size and $Q = Q^T > 0$, the following inequality holds

$$X^T Y + Y^T X \leq X^T Q X + Y^T Q^{-1} Y$$

Property 5 Let A and B be matrices of appropriate dimensions and ranks, with $B = B^T > 0$. Then

$$(A - B)^T B^{-1} (A - B) \geq 0 \Longleftrightarrow A^T B^{-1} A \geq A + A^T - B$$

Many control and estimation problems can be written as a double sum negativity (or positivity) problem

$$\sum_{i=1}^{r} \sum_{j=1}^{r} h_i(\boldsymbol{z}(k)) h_j(\boldsymbol{z}(k)) \Gamma_{ij}(\boldsymbol{x}) < 0, \tag{2.5}$$

with the symmetric matrices $\Gamma_{ij}(\boldsymbol{x})$ being affinely dependent on the unknown variables $\boldsymbol{x} \in \mathbb{R}^{n_x}$ and the membership functions $h_i(\boldsymbol{z}(k))$ having the convex sum property, i.e., $\sum_{i=1}^{r} h_i(\boldsymbol{z}(k)) = 1$ and $h_i(\boldsymbol{z}(k)) \geq 0$. Note that in what follows, our goal is to obtain such double sums.

The aim is to find conditions on Γ_{ij} such that (2.5) holds, generally using only the convex sum property for the nonlinear functions $h_i(\boldsymbol{z}(k))$. The trivial LMI solution of the problem (2.5) is: $\Gamma_{ij}(\boldsymbol{x}) < 0$, $i, j = 1, \ldots, r$. These conditions can be relaxed by considering that $h_i(\boldsymbol{z}(k)) \geq 0$ and $h_i(\boldsymbol{z}(k)) h_j(\boldsymbol{z}(k)) = h_j(\boldsymbol{z}(k)) h_i(\boldsymbol{z}(k))$. For the ease of notation, in what follows, the dependence of Γ on $\boldsymbol{x}$ is omitted. A basic sufficient solution is (Wang et al., 1996)

Lemma 2.10 *(Wang et al., 1996) Equation* (2.5) *is satisfied if*

$$\begin{aligned} &\Gamma_{ii} < 0 \\ &\Gamma_{ij} + \Gamma_{ji} < 0 \end{aligned} \tag{2.6}$$

for $i = 1, 2, \ldots, r,\ j = i+1, i+2, \ldots, r.$

A refinement of the conditions (2.6) is:

Lemma 2.11 *(Tuan et al., 2001) Equation* (2.5) *is satisfied if*

$$\begin{aligned} &\Gamma_{ii} < 0 \\ &\frac{2}{r-1}\Gamma_{ii} + \Gamma_{ij} + \Gamma_{ji} < 0 \end{aligned} \tag{2.7}$$

for $i = 1, 2, \ldots, r,\ j = 1, 2, \ldots, r,\ i \neq j.$

Adding auxiliary variables can also be useful in order to reduce the conservatism of the conditions.

Lemma 2.12 *(Liu and Zhang, 2003) Condition* (2.5) *is satisfied if there exist matrices* $Q_{ii} > 0$, $i = 1, 2, \ldots, r$, *and* $Q_{ij} = Q_{ji}^T$, $i = 1, 2, \ldots, r,\ j = i+1, i+2, \ldots, r$ *such that*

$$\begin{aligned} &\Gamma_{ii} + Q_{ii} < 0 \\ &\Gamma_{ij} + \Gamma_{ji} + Q_{ij} + Q_{ji} < 0 \\ &\begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1r} \\ Q_{21} & Q_{22} & \cdots & Q_{2r} \\ \vdots & & \ddots & \vdots \\ Q_{r1} & Q_{r2} & \cdots & Q_{rr} \end{pmatrix} > 0 \end{aligned} \tag{2.8}$$

for $i = 1, 2, \ldots, r,\ j = i+1, i+2, \ldots, r.$

The conditions of Lemmas 2.10, 2.11, and 2.12 are only sufficient. Nevertheless, some relaxations exist that become asymptotically necessary when the number of terms in the summations is extended to infinity; Sala and Ariño (2007) proposed conditions that are based on Polya's theorems, whereas Kruszewski et al. (2009) proposed conditions that are based on triangulation. Other works use more properties of the nonlinear functions $h_i(z)$, $i = 1, 2, \ldots, r$, (Sala and Ariño, 2007; Bernal et al., 2009). The main drawback of these results is that the complexity of the LMI problems increases, and they quickly become intractable for the actual LMI solvers.

Many simple and relaxed conditions can be obtained using Finsler's lemma. In fact, most state-of-the-art results in the discrete-time case have been obtained using this lemma. We introduce it here as it will be frequently used in what follows.

Lemma 2.13 *(Skelton et al., 1998) Consider a vector* $\boldsymbol{x} \in \mathbb{R}^{n_x}$ *and two matrices* $Q = Q^T \in \mathbb{R}^{n_x \times n_x}$ *and* $R \in \mathbb{R}^{m \times n_x}$ *such that* $rank(R) < n_x$. *The two following expressions are equivalent:*

1. $\boldsymbol{x}^T Q \boldsymbol{x} < 0$, $\boldsymbol{x} \in \{\boldsymbol{x} \in \mathbb{R}^{n_x}, \boldsymbol{x} \neq 0, R\boldsymbol{x} = 0\}$
2. $\exists M \in \mathbb{R}^{m \times n_x}$ *such that* $Q + MR + R^T M^T < 0$

In what follows, we present some basic results on stability analysis of discrete-time Takagi-Sugeno models of the form

$$\begin{aligned} \boldsymbol{x}(k+1) &= \sum_{i=1}^{r} h_i(\boldsymbol{z}_k) A_i \boldsymbol{x}(k) \\ &= A_z \boldsymbol{x}(k) \end{aligned} \tag{2.9}$$

2.4 Stability analysis – the basics

In order to analyze the stability of a TS model, the direct Lyapunov approach has been used. Stability conditions have been derived using quadratic Lyapunov functions (Tanaka et al., 1998; Tanaka and Wang, 2001; Sala et al., 2005), piecewise continuous Lyapunov functions (Johansson et al., 1999; Feng, 2004), and more recently, to reduce the conservativeness of the conditions, nonquadratic Lyapunov functions (Guerra and Vermeiren, 2004; Kruszewski et al., 2008; Mozelli et al., 2009). The aim is generally to develop conditions in the form of LMIs.

The Lyapunov function[1] classically used is the quadratic one,

$$V(\boldsymbol{x}(k)) = \boldsymbol{x}^T(k) P \boldsymbol{x}(k) \tag{2.10}$$

with $P = P^T > 0$.

Since this Lyapunov function is quadratic in $\boldsymbol{x}(k)$, we speak of "quadratic stability". Note that when a system is quadratically stable it *implies* that it is stable. However, the reverse is *not necessarily* true. Therefore, conditions obtained using the Lyapunov function (2.10) are only sufficient, i.e., if they fail, nothing can be directly said about stability or instability of the TS model.

The TS model (2.9) is quadratically stable if the Lyapunov function (2.10) decreases and tends to zero when $k \to \infty$ for all trajectories $\boldsymbol{x}_k$.

A very first result has been expressed as:

Theorem 2.14 *(Tanaka and Wang, 2001) The unforced model*

$$\boldsymbol{x}(k+1) = A_z \boldsymbol{x}(k)$$

[1] In the sequel, whenever it is evident, the explicit dependence of the Lyapunov function on the state variables is omitted.

is globally asymptotically stable if there exist a matrix $P = P^T$ such that the following LMI problem is feasible

$$A_i^T P A_i - P < 0 \tag{2.11}$$

for $i = 1, 2, \ldots, r$.

The above condition, using the Schur complement, is equivalent to

$$\begin{pmatrix} P & A_i^T P \\ PA_i & P \end{pmatrix} > 0 \tag{2.12}$$

which is its most frequently encountered form.

Quadratic stability of discrete-time TS systems ensures exponential decay of the state values. To verify whether this decay-rate is more than a certain value or indeed to find an upper bound on the decay rate, one can use the following result (Tanaka and Wang, 2001):

Theorem 2.15 *(Tanaka and Wang, 2001) The decay-rate of the unforced model $\boldsymbol{x}(k+1) = A_z \boldsymbol{x}(k)$ is at least $\beta \in [0, 1]$ if the following LMI problem is feasible*

$$\begin{pmatrix} \beta P & A_i^T P \\ PA_i & P \end{pmatrix} > 0 \tag{2.13}$$

for $i = 1, 2, \ldots, r$.

Finding the maximum decay rate is a GEVP problem, which can be solved iteratively.

Since most results in this thesis are based on using Lemma 2.13, here we show how a similar result can be obtained. Consider the Lyapunov function $V = \boldsymbol{x}(k)^T X \boldsymbol{x}(k)$. The change in the Lyapunov functions along the trajectories of the system is:

$$\begin{aligned} \Delta V &= \boldsymbol{x}(k+1)^T X \boldsymbol{x}(k+1) - \boldsymbol{x}(k)^T X \boldsymbol{x}(k) \\ &= \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -X & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k) \end{pmatrix} \end{aligned}$$

At the same time, the dynamics of the discrete-time system can be written as

$$\begin{pmatrix} A_z & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} = 0$$

Using Lemma 2.13 and choosing $M = \begin{pmatrix} 0 \\ H \end{pmatrix}$, with H being a free matrix, one obtains the conditions:

$$\begin{pmatrix} -X & 0 \\ 0 & X \end{pmatrix} + \begin{pmatrix} 0 \\ H \end{pmatrix} \begin{pmatrix} A_z & -I \end{pmatrix} + (*) < 0$$

leading to

$$\begin{pmatrix} -X & (*) \\ HA_z & X-H-H^T \end{pmatrix} < 0$$

A sufficient LMI condition can be formulated as

$$\begin{pmatrix} -X & (*) \\ HA_i & X-H-H^T \end{pmatrix} < 0 \qquad i = 1,2,\ldots,r$$

This result is less conservative than that in Theorem 2.14, due to the free matrix H. To see this, lets us chose $H = H^T = X = P$. With this choice, we have

$$\begin{pmatrix} -P & (*) \\ PA_z & -P \end{pmatrix} < 0$$

for which the LMIs are those in (2.12).

The conditions above are only sufficient conditions, due to using only the knowledge of the convex sum property for $h_i(\boldsymbol{z})$ and reducing the stability issue to quadratic stability. Indeed, a system may be stable but not quadratically stable, see e.g., (Johansson and Rantzer, 1998).

Introducing some "knowledge" in the Lyapunov function can relax the stability conditions. Since, in a sense, a TS model induces a state space partition according to the scheduling variables, current results rely on introducing this partition into the Lyapunov function. This can be done in two ways.

The first way is the use of piecewise quadratic Lyapunov functions (Johansson and Rantzer, 1998; Johansson et al., 1999; Feng, 2003, 2006). The state space is partitioned according to the activation of the linear models, allowing the Lyapunov function to change from one region to another, for instance

$$V(\boldsymbol{x},\boldsymbol{z}) = \boldsymbol{x}^T P(\boldsymbol{z})\boldsymbol{x} \quad P(\boldsymbol{z}) = P_i > 0 \quad \text{if } \boldsymbol{z} \in S_i$$

where S_i are given sets such that $\bigcup_i S_i$ covers the state space. For example, "natural" regions for TS models are: $S_i = \{\boldsymbol{z} | h_i(\boldsymbol{z}) \geq h_j(\boldsymbol{z}),\ j = 1,2,\ldots,r,\ j \neq i\}$, $i = 1,2,\ldots,r$.

The above partition is natural for those TS models that do not have all their linear models activated at once. Unfortunately, this assumption does not hold for TS models built by using the sector nonlinearity approach, but it holds e.g., for TS models obtained by dynamic linearization or substitution.

The second way is to use a fuzzy Lyapunov function (Blanco et al., 2001; Tanaka et al., 2003). The functions

$$\begin{aligned} V(\boldsymbol{x},\boldsymbol{z}) &= \boldsymbol{x}^T \sum_{i=1}^{r} h_i(\boldsymbol{z}) P_i \boldsymbol{x} \\ &= \boldsymbol{x}^T P_z \boldsymbol{x} \end{aligned} \tag{2.14}$$

or

$$\begin{aligned} V(\boldsymbol{x},\boldsymbol{z}) &= \boldsymbol{x}^T \left(\sum_{i=1}^{r} h_i(\boldsymbol{z}) P_i\right)^{-1} \boldsymbol{x} \\ &= \boldsymbol{x}^T P_z^{-1} \boldsymbol{x} \end{aligned} \tag{2.15}$$

with $P_i > 0$, $i = 1, 2, \ldots, r.$ are usually considered, with $P_i > 0$, $i = 1, 2, \ldots, r$, thus introducing the nonlinear membership functions $h_i(\boldsymbol{z})$ into the Lyapunov function.

While in the continuous-time case the main challenge is handling the derivative of the membership functions, in the discrete-time case the derivatives are conveniently avoided. To see this, consider the discrete-time TS model

$$\boldsymbol{x}(k+1) = A_z \boldsymbol{x}(k) \tag{2.16}$$

and the nonquadratic Lyapunov function

$$V = \boldsymbol{x}(k)^T P_z \boldsymbol{x}(k) \tag{2.17}$$

The difference in the Lyapunov function is

$$\begin{aligned} \Delta V &= \boldsymbol{x}(k+1)^T P_{z+1} \boldsymbol{x}(k+1) - \boldsymbol{x}(k)^T P_z \boldsymbol{x}(k) \\ &= \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -P_z & 0 \\ 0 & P_{z+1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} \end{aligned}$$

At the same time, the dynamics of the system can be written as

$$\begin{pmatrix} A_z & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} = 0$$

Using Lemma 2.13, similarly to the quadratic case, and choosing $M = \begin{pmatrix} 0 \\ H_z \end{pmatrix}$ we obtain

$$\begin{pmatrix} -P_z & 0 \\ 0 & P_{z+1} \end{pmatrix} + \begin{pmatrix} 0 \\ H_z \end{pmatrix} \begin{pmatrix} A_z & -I \end{pmatrix} + (*) < 0$$

$$\begin{pmatrix} -P_z & (*) \\ H_z A_z & -H_z - H_z^T + P_{z+1} \end{pmatrix} < 0$$

This result has been formulated as (Guerra and Vermeiren, 2004):

Theorem 2.16 *The TS model* (2.16) *is asymptotically stable if there exist* $P_i = P_i^T > 0$, H_i *so that*

$$\begin{pmatrix} -P_z & (*) \\ H_z A_z & -H_z - H_z^T + P_{z+1} \end{pmatrix} < 0$$

Conveniently, sufficient LMI conditions can be formulated using e.g., Lemma 2.11 as follows:

Corollary 2.17 *The TS model* (2.16) *is asymptotically stable if there exist* $P_i = P_i^T > 0$*,* H_i *so that*

$$\frac{2}{r-1}\Gamma_{i,j}^k + \Gamma_{i,j}^k + \Gamma_{j,i}^k < 0$$

$i, j, k = 1, 2, \ldots, r$*, where*

$$\Gamma_{i,j}^k = \begin{pmatrix} -P_i & (*) \\ H_i A_j & -H_i - H_i^T + P_k \end{pmatrix}$$

Naturally, several other relaxations, such as those presented in (Wang and Mendel, 1992), (Wang et al., 1996), (Sala and Ariño, 2007) can also be used.

Another possibility is to use the inverse of the convex sum in the Lyapunov function, i.e., (2.15). In this case the difference in the Lyapunov function is

$$\begin{aligned} \Delta V &= \boldsymbol{x}(k+1)^T P_{z+1}^{-1} \boldsymbol{x}(k+1) - \boldsymbol{x}(k)^T P_z^{-1} \boldsymbol{x}(k) \\ &= \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -P_z^{-1} & 0 \\ 0 & P_{z+1}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} \end{aligned}$$

The dynamics of the system are

$$\begin{pmatrix} A_z & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} = 0$$

Similarly to the previous case, using Lemma 2.13, and choosing $M = \begin{pmatrix} 0 \\ P_{z+1}^{-1} \end{pmatrix}$ we obtain

$$\begin{pmatrix} -P_z^{-1} & 0 \\ 0 & P_{z+1}^{-1} \end{pmatrix} + \begin{pmatrix} 0 \\ P_{z+1}^{-1} \end{pmatrix} \begin{pmatrix} A_z & -I \end{pmatrix} + (*) < 0$$

$$\begin{pmatrix} -P_z^{-1} & (*) \\ P_{z+1}^{-1} A_z & -P_{z+1}^{-1} \end{pmatrix} < 0$$

Congruence with $\begin{pmatrix} H_z^T & 0 \\ 0 & P_{z+1} \end{pmatrix}$ leads to

$$\begin{pmatrix} -H_z^T P_z^{-1} H_z & (*) \\ A_z H_z & -P_{z+1} \end{pmatrix} < 0$$

which, by applying Property 5, yields

$$\begin{pmatrix} -H_z^T + P_z - H_z & (*) \\ A_z H_z & -P_{z+1} \end{pmatrix} < 0$$

for which again sufficient LMI conditions can be formulated and relaxations can be used.

This result can be formulated as

Theorem 2.18 *The TS model* (2.16) *is asymptotically stable if there exist* $P_i = P_i^T > 0$*,* H_i *so that*

$$\begin{pmatrix} -H_z^T + P_z - H_z & (*) \\ A_z H_z & -P_{z+1} \end{pmatrix} < 0$$

A different result, which can actually be applied to any Lyapunov function in the discrete-time case, relies on considering the variation of the Lyapunov function during several samples (Kruszewski et al., 2008). Up to this point, we imposed that the Lyapunov function decreases every sample. However, for a stable system, the states will eventually converge to zero, even though the Lyapunov function does not decrease every time instant, but only after several samples. This is why this technique is called α-sample variation.

Consider again the Lyapunov function (2.17). In this case, we analyze the variation of the function after α samples, which can be written as

$$\begin{aligned} \Delta_\alpha V &= \boldsymbol{x}(k+\alpha)^T P_{z+\alpha} \boldsymbol{x}(k+\alpha) - \boldsymbol{x}(k)^T P_z \boldsymbol{x}(k) \\ &= \begin{pmatrix} \boldsymbol{x}(k) \\ \vdots \\ \boldsymbol{x}(k+\alpha) \end{pmatrix}^T \begin{pmatrix} -P_z & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \vdots & P_{z+\alpha} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \vdots \\ \boldsymbol{x}(k+\alpha) \end{pmatrix} \end{aligned}$$

This result has been formulated as (Kruszewski et al., 2008):

Theorem 2.19 *The TS model* (2.16) *is asymptotically stable if there exist* $P_i = P_i^T > 0$*,* H_i *so that*

$$\begin{pmatrix} -P_z & (*) & (*) & \dots & (*) \\ H_z A_z & -H_z - (*) & (*) & \dots & (*) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & -H_{z+\alpha-1} - (*) + P_{z+\alpha} \end{pmatrix} < 0$$

Similarly to the previous results, sufficient LMI conditions can be formulated and all the existing relaxations can be used.

It has to be noted, that to simply prove stability, in general, the membership functions and their properties are not needed or used. Moreover, relaxations such as Lemma 2.11 or 2.12 can easily used, or, due to the double sum, even Polya's theorem (Sala and Ariño, 2007) can be applied, obtaining asymptotically necessary and sufficient conditions.

However, since the TS model is usually defined on a compact set, the region in which stability holds may be very restricted. Therefore, to also obtain the stability region, the properties of the scheduling variables are used. For instance, in the continuous case, given a bound on the derivatives of the membership functions, the result in (Mozelli et al., 2009) or if the TS model has been obtained by using the sector nonlinearity approach, the results in (Bernal and Guerra, 2010) can be directly employed. In the discrete-time case, we have developed such results and will present them in Part III.

In the last years, next to using the single-sum Lyapunov functions shown above, double- and n-sum Lyapunov functions have also been introduced, together with delayed Lyapunov function. While they can also be used to determine stability of the system, the interest in using them will be shown for stabilization in the next chapter.

Chapter 3

Nonquadratic stabilization

3.1 Introduction

Non-quadratic Lyapunov functions have significantly improved the design conditions in the discrete-time case (Guerra and Vermeiren, 2004; Ding et al., 2006; Dong and Yang, 2009; Lee et al., 2011). The solutions obtained by non-quadratic Lyapunov functions include and extend the set of solutions obtained using the quadratic framework. One-sum Lyapunov functions have quickly been extended to double-sum Lyapunov functions in (Ding et al., 2006) and later to polynomial Lyapunov functions in (Sala and Ariño, 2007; Ding, 2010; Lee et al., 2010).

In the nonquadratic framework, delayed controllers and observers have been proposed in (Kerkeni et al., 2010). The observer design method has been generalized further on in (Guerra et al., 2012), but the controller design had the shortcoming of an increased number of LMIs.

In what follows, we present a general framework for using delayed non-quadratic Lyapunov functions for controller design, and new possibilities thanks to the delayed aspect of the Lyapunov function. These allow the use of a completely new family of controllers based on past states. Moreover, the results are extended for robust control, H_∞ control and to α-sample variation.

The material in this chapter is based on the following publications:

(P1) Zs. Lendek, T. M. Guerra, J. Lauber, Controller design for TS models using non-quadratic Lyapunov functions. IEEE Transactions on Cybernetics, vol. 45, no. 3, pages 453-464, 2015.

(P2) Zs. Lendek, T.M. Guerra, J. Lauber, Construction of extended Lyapunov functions and control laws for discrete-time TS systems. *Proceedings of the 2012 IEEE World Congress on Computational Intelligence, IEEE International Conference on Fuzzy Systems*, pages 286-291, Brisbane, Australia, June 2012.

3.2 A motivating example

To show the interest in using delayed states in the Lyapunov function and in the controller gains, we first present a simple example.

Thus, consider the TS model

$$\begin{aligned} \boldsymbol{x}(k+1) &= \sum_{i=1}^{r} h_i(\boldsymbol{z}(k))(A_i\boldsymbol{x}(k) + B_i\boldsymbol{u}(k)) \\ &= A_z\boldsymbol{x}(k) + B_z\boldsymbol{u}(k) \end{aligned} \tag{3.1}$$

and the controller

$$\begin{aligned} \boldsymbol{u}(k) = &-\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\boldsymbol{z}(k))h_j(\boldsymbol{z}(k-1))F_{ij}\cdot \\ &\cdot\left(\sum_{l=1}^{r}\sum_{m=1}^{r} h_l(\boldsymbol{z}(k))h_m(\boldsymbol{z}(k-1))H_{lm}\right)^{-1}\boldsymbol{x}(k) \\ = &-F_{zz-}H_{zz-}^{-1}\boldsymbol{x}(k) \end{aligned} \tag{3.2}$$

i.e., the controller gains F and H depend not only on the current state, but also the past one.

The closed-loop dynamics are:

$$\boldsymbol{x}(k+1) = \left(A_z - B_zF_{zz-}H_{zz-}^{-1}\right)\boldsymbol{x}(k) \tag{3.3}$$

Using the Lyapunov function $V = \boldsymbol{x}^T(k)P_{z-}^{-1}\boldsymbol{x}(k)$, we have the difference

$$\begin{aligned} \Delta V_1 =& V(k+1) - V(k) = \\ =& \boldsymbol{x}(k+1)^T P_z^{-1}\boldsymbol{x}(k+1) - \boldsymbol{x}(k)^T P_{z-}^{-1}\boldsymbol{x}(k) \\ =& \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -P_{z-}^{-1} & 0 \\ 0 & P_z^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} \end{aligned}$$

The closed-loop system dynamic is

$$\begin{pmatrix} A_z - B_zF_{zz-}H_{zz-}^{-1} & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} = 0 \tag{3.4}$$

Using Lemma 2.13 with (3.4), $\Delta V_1 < 0$, if there exist $M \in \mathbb{R}^{2n_x \times n_x}$ so that

$$\begin{pmatrix} -P_{z-}^{-1} & 0 \\ 0 & P_z^{-1} \end{pmatrix} + M\begin{pmatrix} A_z - B_zF_{zz-}H_{zz-}^{-1} & -I \end{pmatrix} + (*) < 0$$

Choosing $M = \begin{pmatrix} 0 \\ P_z^{-1} \end{pmatrix}$ and congruence with $\begin{pmatrix} H_{zz-} & 0 \\ 0 & P_z \end{pmatrix}$ leads to

$$\begin{pmatrix} -H_{zz-}^T P_{z-}^{-1} H_{zz-} & (*) \\ A_zH_{zz-} - B_zF_{zz-} & -P_z \end{pmatrix} < 0$$

Using Property 5, we obtain

$$\begin{pmatrix} -H_{zz-} - H_{zz-}^T + P_{z-} & (*) \\ A_z H_{zz-} - B_z F_{zz-} & -P_z \end{pmatrix} < 0 \tag{3.5}$$

This result can be formulated as

Theorem 3.1 *The closed-loop dynamics* (3.3) *are asymptotically stable if there exist* $P_z = P_z^T$, F_{zz-}, *and* H_{zz-}, *so that* (3.5) *holds.*

Relaxed LMI conditions can easily be formulated using Lemmas 2.10 or 2.11, as follows.

Corollary 3.2 *The closed-loop system* (3.3) *is asymptotically stable, if there exist* $P_i = P_i^T$, F_{ij}, *and* H_{ij}, $i, j = 1, 2, \ldots, r$ *so that*

$$\Gamma_{ijk} + \Gamma_{jik} < 0$$

for $i, j, k = 1, 2, \ldots, r$, $i \leq j$, *or*

$$\Gamma_{iik} < 0$$
$$\frac{2}{r-1}\Gamma_{iik} + \Gamma_{ijk} + \Gamma_{jik} < 0$$

for $i, j, k = 1, 2, \ldots, r$, *where*

$$\Gamma_{ijk} = \begin{pmatrix} -H_{jk} - H_{jk}^T + P_k & (*) \\ A_i H_{jk} - B_i F_{jk} & -P_i \end{pmatrix} < 0$$

The proof of the above corollary is straightforward.

Note that the conditions above are not equivalent to those in the literature, e.g., those in (Guerra and Vermeiren, 2004), which involve the negative definiteness of sums of the form

$$\begin{pmatrix} -P_z & (*) \\ A_z H_z - B_z F_z & -H_{z+} - H_{z+}^T + P_{z+} \end{pmatrix} < 0 \tag{3.6}$$

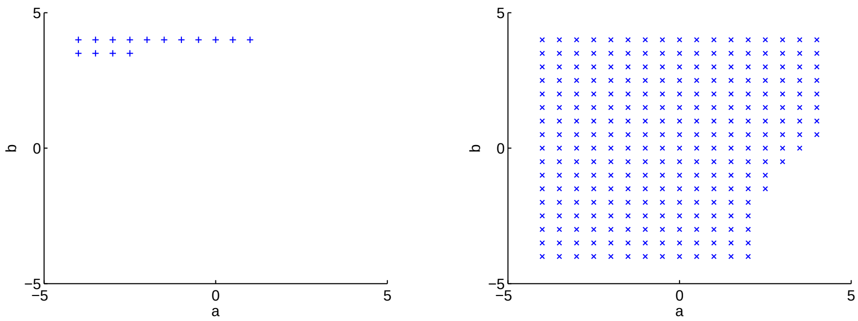
This will be shown in the following example.

Remark: Both using P_{z-} in the Lyapunov function and the controller gain $F_{zz-}H_{zz-}$ and P_z in the Lyapunov function and the controller gain $F_z H_z$, as in (3.6), respectively, result in r^3 LMIs.

Example 3.1 Consider the two-rule TS model having the local matrices

$$A_1 = \begin{pmatrix} 2 & 0.04a + 6.9 \\ -1 & 0.03b - 2.9 \end{pmatrix} \quad B_1 = \begin{pmatrix} 0.03b - 2.9 \\ 1 \end{pmatrix}$$
$$A_2 = \begin{pmatrix} 1 & 0.04a + 6.9 \\ -1 & 0.03b - 2.9 \end{pmatrix} \quad B_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

where a and b are real-valued parameters, $a, b \in [-4, 4]$. Using the conditions in (Guerra and Vermeiren, 2004), the values of a and b for which a solution can be found are presented in Figure 3.1(a). By using the conditions of Theorem 3.1, we obtain solutions for the values presented in Figure 3.1(b). In both cases, the number of sums involved is 3. □



(a) Results using the conditions in Guerra and Vermeiren (2004).

(b) Results using Theorem 3.1.

Figure 3.1: Feasible solutions for Example 3.1.

Note that an equivalence between the methods above would correspond to a controller gain $F_{zz+}H_{zz+}$, which cannot be used. Therefore, considering P_{z-} instead of P_z allows introducing more degrees of freedom in the control law.

We will revisit this example later on. The question remains, that if more sums are used in P, F, and H how and which sums should be used. The result will be generalized with extra delayed states $z-1$, $z-2$, etc., allowing extended control using past states. In order to do the generalization, we use the notation introduced in Section 2.2.

3.3 Two controller design methods

In what follows, the discrete-time TS model considered for controller design is of the form

$$\boldsymbol{x}(k+1) = A_z \boldsymbol{x}(k) + B_z \boldsymbol{u}(k) \tag{3.7}$$

where A_z denotes the convex sum $A_z = \sum_{i=1}^{r} h_i(\boldsymbol{z}(k))A_i$, A_i and B_i, $i = 1, 2, \ldots, r$ are the local matrices, r denotes the number of rules, k is the time instant, $\boldsymbol{x} \in \mathbb{R}^{n_x}$ is the state vector, $\boldsymbol{u} \in \mathbb{R}^{n_u}$ is the control input, $\boldsymbol{z} \in \mathbb{R}^{n_z}$ is the scheduling vector. It is assumed that the scheduling variables $\boldsymbol{z}(k)$ are available at the time instant k.

With the notations defined in Section 2.2, the system (3.7) can be written as

$$\boldsymbol{x}(k+1) = \mathbb{A}_{G_0^A} \boldsymbol{x}(k) + \mathbb{B}_{G_0^B} \boldsymbol{u}(k) \tag{3.8}$$

with $G_0^A = G_0^B = \{0\}$. Although we restrict ourselves to the classical TS system of the form (3.7), for the ease of notation we use the notation (3.8).

The controller used is of the form

$$\boldsymbol{u}(k) = -\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}\boldsymbol{x}(k) \tag{3.9}$$

with $\mathbb{F}_{G_0^F}$ and $\mathbb{H}_{G_0^H}$ being multiple sums with delays given by G_0^F, $|G_0^F| = n_F$, and G_0^H, $|G_0^H| = n_H$, respectively. Note that G_0^F and G_0^H may not contain positive delays, since a positive delay refers to future scheduling variables, that are not available. At this point, the possible delays, i.e., the multisets G_0^F and G_0^H are not necessarily determined. Possible delays will be discussed in Section 3.3.2. Note that this controller is a generalization of those used in (Kerkeni et al., 2009; Sala and Ariño, 2007; Guerra and Vermeiren, 2004).

Using the controller (3.9) for the TS system (3.8), the closed-loop system can be expressed as

$$\boldsymbol{x}(k+1) = \left(\mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}\right)\boldsymbol{x}(k) \tag{3.10}$$

3.3.1 Design conditions

Conditions have been developed using two different Lyapunov functions:

- **Case 1:** $V = \boldsymbol{x}(k)^T\mathbb{H}_{G_0^H}^{-T}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1}\boldsymbol{x}(k)$, with $P_{\boldsymbol{i}} = P_{\boldsymbol{i}}^T > 0$, for $\boldsymbol{i} \in \mathbb{I}_{G_0^P}$, $|G_0^P| = n_P$, and $\mathbb{H}_{G_0^H}$ being the multisum used in the controller, and

- **Case 2:** $V = \boldsymbol{x}(k)^T\mathbb{P}_{G_0^P}^{-1}\boldsymbol{x}(k)$, with $P_{\boldsymbol{i}} = P_{\boldsymbol{i}}^T > 0$, for $\boldsymbol{i} \in \mathbb{I}_{G_0^P}$, $|G_0^P| = n_P$.

In what follows, whenever referring to Case 1 and Case 2, we refer to the two Lyapunov functions above.

For Case 1, the following result can be stated:

Theorem 3.3 *The closed-loop system* (3.10) *is asymptotically stable, if there exist* $P_{\boldsymbol{i}_j^P} = P_{\boldsymbol{i}_j^P}^T$, $\boldsymbol{i}_j^P = pr_{G_j^P}^{\boldsymbol{i}}$, *and* $H_{\boldsymbol{i}_j^H}$, $\boldsymbol{i}_j^H = pr_{G_j^H}^{\boldsymbol{i}}$, $\boldsymbol{i} \in \mathbb{I}_{G_V}$, $j = 0, 1$, *and* $F_{\boldsymbol{i}_0^F}$, $\boldsymbol{i}_0^F = pr_{G_0^F}^{\boldsymbol{i}}$, *where* $G_V = G_0^P \cup G_1^P \cup (G_0^F \oplus G_0^B) \cup (G_0^H \oplus G_0^A) \cup G_1^H$ *so that*

$$\begin{pmatrix} -\mathbb{P}_{G_0^P} & (*) \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & -\mathbb{H}_{G_1^H} - \mathbb{H}_{G_1^H}^T + \mathbb{P}_{G_1^P} \end{pmatrix} < 0 \tag{3.11}$$

Remark: G_V above is simply the multiset containing all the delays in the multiple sum in (3.11).

Proof: Consider the Lyapunov function $V = \boldsymbol{x}(k)^T \mathbb{H}_{G_0^H}^{-T} \mathbb{P}_{G_0^P} \mathbb{H}_{G_0^H}^{-1} \boldsymbol{x}(k)$, with $P_{\boldsymbol{i}} = P_{\boldsymbol{i}}^T > 0$, for $\boldsymbol{i} \in \mathbb{I}_{G_0^P}$, $|G_0^P| = n_P$. The difference is

$$\begin{aligned}\Delta V_1 =& V(k+1) - V(k) = \\ =& \boldsymbol{x}(k+1)^T \mathbb{H}_{G_1^H}^{-T} \mathbb{P}_{G_1^P} \mathbb{H}_{G_1^H}^{-1} \boldsymbol{x}(k+1) - \boldsymbol{x}(k)^T \mathbb{H}_{G_0^H}^{-T} \mathbb{P}_{G_0^P} \mathbb{H}_{G_0^H}^{-1} \boldsymbol{x}(k) \\ =& \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -\mathbb{H}_{G_0^H}^{-T} \mathbb{P}_{G_0^P} \mathbb{H}_{G_0^H}^{-1} & 0 \\ 0 & \mathbb{H}_{G_1^H}^{-T} \mathbb{P}_{G_1^P} \mathbb{H}_{G_1^H}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}\end{aligned}$$

The closed-loop system dynamics are

$$\begin{pmatrix} \mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B} \mathbb{F}_{G_0^F} \mathbb{H}_{G_0^H}^{-1} & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} = 0 \tag{3.12}$$

Using Lemma 2.13, $\Delta V_1 < 0$, if there exist $M \in \mathbb{R}^{2n_x \times n_x}$ so that

$$\begin{pmatrix} -\mathbb{H}_{G_0^H}^{-T} \mathbb{P}_{G_0^P} \mathbb{H}_{G_0^H}^{-1} & 0 \\ 0 & \mathbb{H}_{G_1^H}^{-T} \mathbb{P}_{G_1^P} \mathbb{H}_{G_1^H}^{-1} \end{pmatrix} + M \begin{pmatrix} \mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B} \mathbb{F}_{G_0^F} \mathbb{H}_{G_0^H}^{-1} & -I \end{pmatrix} + (*) < 0$$

In order to obtain a problem with LMI constraints encompassing the classical cases, a choice is:

$$M = \begin{pmatrix} 0 \\ \mathbb{H}_{G_1^H}^{-T} \end{pmatrix}$$

which leads to

$$\begin{pmatrix} -\mathbb{H}_{G_0^H}^{-T} \mathbb{P}_{G_0^P} \mathbb{H}_{G_0^H}^{-1} & (*) \\ \mathbb{H}_{G_1^H}^{-T} \mathbb{A}_{G_0^A} - \mathbb{H}_{G_1^H}^{-T} \mathbb{B}_{G_0^B} \mathbb{F}_{G_0^F} \mathbb{H}_{G_0^H}^{-1} & -\mathbb{H}_{G_1^H}^{-T} - \mathbb{H}_{G_1^H}^{-1} + \mathbb{H}_{G_1^H}^{-T} \mathbb{P}_{G_1^P} \mathbb{H}_{G_1^H}^{-1} \end{pmatrix} < 0 \tag{3.13}$$

Applying to (3.13) Property 1 with the matrix

$$\begin{pmatrix} \mathbb{H}_{G_0^H}^T & 0 \\ 0 & \mathbb{H}_{G_1^H}^T \end{pmatrix}$$

gives directly the conditions (3.11). ■

Case 2 leads to the conditions:

Theorem 3.4 *The closed-loop system* (3.10) *is asymptotically stable, if there exist* $P_{\boldsymbol{i}_j^P} = P_{\boldsymbol{i}_j^P}^T$, $\boldsymbol{i}_j^P = pr_{G_j^P}^{\boldsymbol{i}}$, $j = 0, 1$, $F_{\boldsymbol{i}_0^F}$, $\boldsymbol{i}_0^F = pr_{G_0^F}^{\boldsymbol{i}}$, *and* $H_{\boldsymbol{i}_0^H}$, $\boldsymbol{i}_0^H = pr_{G_0^H}^{\boldsymbol{i}}$, $\boldsymbol{i} \in \mathbb{I}_{G_V}$, *where* $G_V = G_0^P \cup G_1^P \cup (G_0^F \oplus G_0^B) \cup (G_0^H \oplus G_0^A)$ *so that*

$$\begin{pmatrix} -\mathbb{H}_{G_0^H} - \mathbb{H}_{G_0^H}^T + \mathbb{P}_{G_0^P} & (*) \\ \mathbb{A}_{G_0^A} \mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B} \mathbb{F}_{G_0^F} & -\mathbb{P}_{G_1^P} \end{pmatrix} < 0 \tag{3.14}$$

Proof: Consider the Lyapunov function $V = \boldsymbol{x}(k)^T \mathbb{P}_{G_0^P}^{-1} \boldsymbol{x}(k)$, with $P_{\boldsymbol{i}} = P_{\boldsymbol{i}}^T > 0$, for $\boldsymbol{i} \in \mathbb{I}_{G_0^P}$, $|G_0^P| = n_P$. The difference is

$$\Delta V_1 = \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -\mathbb{P}_{G_0^P}^{-1} & 0 \\ 0 & \mathbb{P}_{G_1^P}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}$$

Using Lemma 2.13 with (3.12), $\Delta V_1 < 0$, if there exist $M \in \mathbb{R}^{2n_x \times n_x}$ so that

$$\begin{pmatrix} -\mathbb{P}_{G_0^P}^{-1} & 0 \\ 0 & \mathbb{P}_{G_1^P}^{-1} \end{pmatrix} + M \begin{pmatrix} \mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B} \mathbb{F}_{G_0^F} \mathbb{H}_{G_0^H}^{-1} & -I \end{pmatrix} + (*) < 0$$

Choosing $M = \begin{pmatrix} 0 \\ \mathbb{P}_{G_1^P}^{-1} \end{pmatrix}$ and congruence with $\begin{pmatrix} \mathbb{H}_{G_0^H} & 0 \\ 0 & \mathbb{P}_{G_1^P} \end{pmatrix}$ leads to

$$\begin{pmatrix} -\mathbb{H}_{G_0^H}^T \mathbb{P}_{G_0^P}^{-1} \mathbb{H}_{G_0^H} & (*) \\ \mathbb{A}_{G_0^A} \mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B} \mathbb{F}_{G_0^F} & -\mathbb{P}_{G_1^P} \end{pmatrix} < 0$$

Using Property 5, we obtain directly (3.14). ■

3.3.2 Examples and discussion

First, we illustrate the use of the conditions (3.11) and (3.14), respectively on the following example. Consider a two-rule fuzzy system

$$\begin{aligned} \boldsymbol{x}(k+1) &= \sum_{i=1}^{2} h_i(\boldsymbol{z}(k))(A_i \boldsymbol{x}(k) + B_i \boldsymbol{u}(k)) \\ &= \mathbb{A}_{G_0^A} \boldsymbol{x}(k) + \mathbb{B}_{G_0^B} \boldsymbol{u}(k) \end{aligned}$$

with $G_0^A = G_0^B = \{0\}$ for which a controller has to be designed and let $G_0^H = \{0, -1\}$, $G_0^F = \{0, -1\}$, $G_0^P = \{-1, -1\}$, i.e.,

$$\begin{aligned} \mathbb{P}_{\{-1,-1\}} &= \sum_{i=1}^{2} \sum_{j=1}^{2} h_i(\boldsymbol{z}(k-1)) h_j(\boldsymbol{z}(k-1)) P_{ij} \\ \mathbb{H}_{\{0,-1\}} &= \sum_{i=1}^{2} \sum_{j=1}^{2} h_i(\boldsymbol{z}(k)) h_j(\boldsymbol{z}(k-1)) H_{ij} \\ \mathbb{F}_{\{0,-1\}} &= \sum_{i=1}^{2} \sum_{j=1}^{2} h_i(\boldsymbol{z}(k)) h_j(\boldsymbol{z}(k-1)) F_{ij} \end{aligned}$$

Then, the conditions (3.11) of Theorem 3.3 correspond to *there exist P_{ij}, F_{ij}, H_{ij}, $i, j = 1, 2$ so that*

$$\begin{pmatrix} -\mathbb{P}_{\{-1,-1\}} & (*) \\ \mathbb{A}_{\{0\}} \mathbb{H}_{\{0,-1\}} - \mathbb{B}_{\{0\}} \mathbb{F}_{\{0,-1\}} & -\mathbb{H}_{\{0,1\}} - \mathbb{H}_{\{0,1\}}^T + \mathbb{P}_{\{0,0\}} \end{pmatrix} < 0$$

or

$$\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\sum_{i_4=1}^{2}\sum_{i_5=1}^{2} h_{i_1}(z(k))h_{i_2}(z(k))h_{i_3}(z(k-1))h_{i_4}(z(k-1))h_{i_5}(z(k+1))\cdot$$
$$\cdot \begin{pmatrix} -P_{i_3 i_4} & (*) \\ A_{i_1}H_{i_2 i_3} - B_{i_1}F_{i_2 i_3} & -H_{i_1 i_5} - H_{i_1 i_5}^T + P_{i_1 i_2} \end{pmatrix} < 0$$

while the conditions (3.14) of Theorem 3.4 correspond to *there exist* P_{ij}, F_{ij}, H_{ij}, $i, j = 1, 2$ *so that*

$$\begin{pmatrix} \mathbb{P}_{\{-1,-1\}} - \mathbb{H}_{\{0,-1\}} - \mathbb{H}_{\{0,-1\}}^T & (*) \\ \mathbb{A}_{\{0\}}\mathbb{H}_{\{0,-1\}} - \mathbb{B}_{\{0\}}\mathbb{F}_{\{0,-1\}} & -\mathbb{P}_{\{0,0\}} \end{pmatrix} < 0$$

or

$$\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\sum_{i_4=1}^{2} h_{i_1}(z(k))h_{i_2}(z(k))h_{i_3}(z(k-1))h_{i_4}(z(k-1))\cdot$$
$$\cdot \begin{pmatrix} -H_{i_2 i_3} - H_{i_2 i_3}^T + P_{i_3 i_4} & (*) \\ A_{i_1}H_{i_2 i_3} - B_{i_1}F_{i_2 i_3} & -P_{i_1 i_2} \end{pmatrix} < 0$$

Assuming classical TS models, the maximum number of sums in (3.11) (Case 1) is given by $|G_V| \le 2n_P + n_F + 2n_H + 1$, which corresponds to the maximum of $|G_0^P \cup G_1^P \cup (G_0^F \oplus G_0^B) \cup (G_0^H \oplus G_0^A) \cup G_1^H|$ or pairwise non-overlapping sets of indices G_0^P, G_1^P, G_0^F, G_0^H, G_1^H, and $\{0\}$. For Case 2, the number of sums in (3.14) is given by $|G_V| \le 2n_P + n_F + n_H + 1$, which in this case corresponds to the maximum of $|G_0^P \cup G_1^P \cup (G_0^F \oplus G_0^B) \cup (G_0^H \oplus G_0^A)|$. Note that when $|G_V| = 2n_P + n_F + 2n_H + 1$ (Case 1) or $|G_V| = 2n_P + n_F + n_H + 1$ (Case 2), the corresponding conditions become equivalent to using a common quadratic Lyapunov function and a PDC controller, as the LMIs have to be solved for each possible index. However, the maximum number of sums indicates that for fixed G_0^H, G_0^F, and G_0^P, in general the conditions of Case 2 will lead to a smaller number of sums and consequently the number of LMIs to be solved.

In general, depending on the exact sets of indices used, G_0^P, G_0^F, and G_0^H, relaxations such as those presented by Wang et al. (1996); Tanaka et al. (1998); Tuan et al. (2001); Sala and Ariño (2007) for the LMIs in (3.11) and (3.14) can be used.

In order to further reduce the number of LMIs and the conservativeness of the conditions, the delays, notably the multisets G_0^P, G_0^H, and G_0^F should be chosen such that relaxations can be used. Since $G_0^A = G_0^B = \{0\}$, both G_0^H, and G_0^F should contain $\{0\}$. It can also be seen that due to the terms $\mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H}$ and $\mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}$, which appear in both cases, one can chose $G_0^F = G_0^H$.

To illustrate the choice of the delays, consider now the simplest case, when $|G_0^P| = 1$, i.e., only one sum is used in P.

For Case 1, we have the inequality

$$\begin{pmatrix} -\mathbb{P}_{G_0^P} & (*) \\ \mathbb{A}_{\{0\}}\mathbb{H}_{\{0\}} - \mathbb{B}_{\{0\}}\mathbb{F}_{\{0\}} & -\mathbb{H}_{\{1\}} - \mathbb{H}_{\{1\}}^T + \mathbb{P}_{G_1^P} \end{pmatrix} < 0$$

which, independent of $|G_0^P|$ already contains 3 sums. By adding another index in G_0^H, the number of sums increases. Moreover, in order to keep this number of sums, $|G_0^P|$ has to be chosen as $|G_0^P| = \{0\}$. For an arbitrary cardinality of the multisets G_0^P and G_0^H, this generalizes to $G_0^P = \{0, 0, \ldots, 0\}$ and $G_0^F = G_0^H = \{0, 0, \ldots, 0\}$. Moreover, if $|G_0^P| = |G_0^H| = n_P$, this choice reduces the number of sums in (3.11) to $2n_P + 1$. This can also be seen from $|G_V|$, which for this case is reduced to $G_V = \{0, \underbrace{0, 0, \ldots, 0}_{n_P}, \underbrace{1, 1, \ldots, 1}_{n_P}\}$.

For Case 2, we have the inequality

$$\begin{pmatrix} -\mathbb{H}_{\{0\}} - \mathbb{H}_{\{0\}}^T + \mathbb{P}_{G_0^P} & (*) \\ \mathbb{A}_{\{0\}}\mathbb{H}_{\{0\}} - \mathbb{B}_{\{0\}}\mathbb{F}_{\{0\}} & -\mathbb{P}_{G_1^P} \end{pmatrix} < 0$$

which contains two sums. In order to have 3 sums (same as in Case 1), G_0^P can be chosen either $\{0\}$ or $\{-1\}$, which would lead to

$$\begin{pmatrix} -\mathbb{H}_{\{0\}} - \mathbb{H}_{\{0\}}^T + \mathbb{P}_{\{0\}} & (*) \\ \mathbb{A}_{\{0\}}\mathbb{H}_{\{0\}} - \mathbb{B}_{\{0\}}\mathbb{F}_{\{0\}} & -\mathbb{P}_{\{1\}} \end{pmatrix} < 0$$

for $G_0^P = \{0\}$ or

$$\begin{pmatrix} -\mathbb{H}_{\{0\}} - \mathbb{H}_{\{0\}}^T + \mathbb{P}_{\{-1\}} & (*) \\ \mathbb{A}_{\{0\}}\mathbb{H}_{\{0\}} - \mathbb{B}_{\{0\}}\mathbb{F}_{\{0\}} & -\mathbb{P}_{\{0\}} \end{pmatrix} < 0$$

for $G_0^P = \{-1\}$, respectively. It can be easily seen that in the second case, i.e., when $G_0^P = \{-1\}$, we can also add[1] another dimension to H and F, that provides more freedom, but without altering the number of sums in the condition. This choice will lead to the conditions

$$\begin{pmatrix} -\mathbb{H}_{\{0,-1\}} - \mathbb{H}_{\{0,-1\}}^T + \mathbb{P}_{\{-1\}} & (*) \\ \mathbb{A}_{\{0\}}\mathbb{H}_{\{0,-1\}} - \mathbb{B}_{\{0\}}\mathbb{F}_{\{0,-1\}} & -\mathbb{P}_{\{0\}} \end{pmatrix} < 0$$

For an arbitrary cardinality of the multisets G_0^P and G_0^H, this choice generalizes to $G_0^P = \{-1, -1, \ldots, -1\}$ and $G_0^F = G_0^H = \{0, 0, \ldots, 0, -1, \ldots, -1\}$. Moreover, if $|G_0^P| = |G_0^H| = 2n_P$, this choice reduces the number of sums in (3.11) to $2n_P + 1$. This can also be seen from $|G_V|$, which, similarly to Case 1, is reduced to $G_V = \{0, \underbrace{0, 0, \ldots, 0}_{n_P}, \underbrace{-1, -1, \ldots, -1}_{n_P}\}$.

[1]Note that $\mathbb{H}_{\{0,1\}}$ cannot be used, as the premise variables are not known in advance.

Theorem 3.3 is a generalization of existing results and in a sense, a way to write them in a convenient general form. Theorem 3.4, by using delayed control, represents a new result that allows bringing new control laws that have not been possible with the previous conditions. To show this, let us look to several results in the literature.

The results of Guerra and Vermeiren (2004) are recovered by using the Lyapunov function from Case 2 with $G_0^P = \{0\}$, $G_0^H = G_0^F = \{0\}$ and choosing $\mathbb{H}_{G^H} = \mathbb{P}_{G^P}$. Theorem 3 of Ding et al. (2006) is obtained from Case 2, by choosing $G_0^P = \{0, 0\}$, and $G_0^H = G_0^F = \{0\}$. The controller design of Dong and Yang (2009) can be recovered from Theorem 3.3. The results of Kerkeni et al. (2009) (without the delay) are again a special case of Theorem 3.3. Theorem 1 of Lee et al. (2010) is Theorem 3.5 for the choice $G_0^P = \{0, \dots, 0\}$, $G_0^H = G_0^F = \{0, \dots, 0\}$. Theorem 1 of Kerkeni et al. (2010) is obtained from Theorem 3.3 of this thesis by choosing $G_0^P = \{-1\}$, and $G_0^H = G_0^F = \{0, -1\}$. The results of Lee et al. (2011) (without the relaxations used on the sums) correspond to Theorem 3.3 applied for the special case of consecutive delays $G_0^P = \{-N+1, -N, \dots, 0, 1\}$, $G_0^H = G_0^F = \{-N_f+1, -N_f, \dots, 0\}$.

Note that the conditions of Theorems 3.3 and 3.4 are not equivalent, and although they generalize several conditions from the literature, they do not include each other. This will be illustrated on two examples. In order to obtain a fair comparison, the delays used are selected as $G_0^F = G_0^H = G_0^P = \{0\}$ for Case 1, and $G_0^F = G_0^H = \{0, -1\}$ and $G_0^P = \{-1\}$ for Case 2. This selection results in 3 sums both for (3.11) and (3.14). On the sums, the relaxation of Wang et al. (1996) is used, and for solving the LMIs, the SeDuMi solver within the Yalmip (Löfberg, 2004) toolbox has been used.

Example 3.2 Consider the two-rule fuzzy system

$$\boldsymbol{x}(k+1) = \sum_{i=1}^{2} h_i(\boldsymbol{z}(k))(A_i\boldsymbol{x}(k) + B_i\boldsymbol{u}(k))$$

with

$$A_1 = \begin{pmatrix} -0.62 & 1.26 \\ 1.44 & -0.35 \end{pmatrix} \quad A_2 = \begin{pmatrix} -1.04 & -0.26 \\ -0.66 & 0.45 \end{pmatrix} \quad B_1 = \begin{pmatrix} -0.73 \\ 1.5 \end{pmatrix} \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For this system, the conditions of Theorem 3.4 are unfeasible, while using Theorem 3.3 we obtain:

$$P_1 = \begin{pmatrix} 66.7315 & -16.9055 \\ -16.9055 & 84.9836 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 63.0577 & 68.0822 \\ 68.0822 & 287.5234 \end{pmatrix}$$
$$H_1 = \begin{pmatrix} 73.2442 & 53.3137 \\ -31.0714 & 96.1056 \end{pmatrix} \qquad H_2 = \begin{pmatrix} 78.2196 & 27.3305 \\ 133.7439 & 278.5545 \end{pmatrix}$$
$$F_1 = \begin{pmatrix} 54.7228 & -20.9555 \end{pmatrix} \qquad F_2 = \begin{pmatrix} -69.5334 & -148.9060 \end{pmatrix}$$

□

Example 3.3 On the other hand, consider the two-rule fuzzy system

$$\boldsymbol{x}(k+1) = \sum_{i=1}^{2} h_i(\boldsymbol{z}(k))(A_i\boldsymbol{x}(k) + B_i\boldsymbol{u}(k))$$

with

$$A_1 = \begin{pmatrix} 1.5 & 2.7 \\ -1.1 & 1.8 \end{pmatrix} \quad A_2 = \begin{pmatrix} -0.4 & -0.8 \\ 0.5 & -0.8 \end{pmatrix} \quad B_1 = \begin{pmatrix} -0.55 \\ 0.9 \end{pmatrix} \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For this system, the conditions of Theorem 3.3 are unfeasible, while using Theorem 3.4 we obtain:

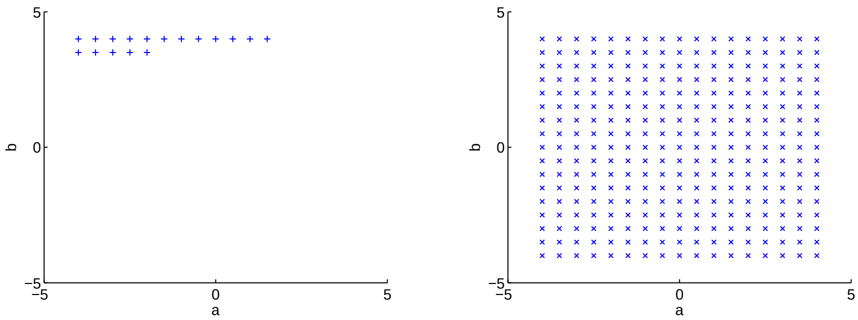
$$P_1 = \begin{pmatrix} 1.9096 & 1.2666 \\ 1.2666 & 1.0716 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 1.0724 & 0.4041 \\ 0.4041 & 0.3692 \end{pmatrix}$$
$$H_{11} = \begin{pmatrix} 1.4702 & 1.0045 \\ 1.0189 & 0.9424 \end{pmatrix} \qquad H_{12} = \begin{pmatrix} 1.8647 & 0.8750 \\ 1.2046 & 0.9967 \end{pmatrix}$$
$$H_{21} = \begin{pmatrix} 0.9167 & 0.3108 \\ 0.3424 & 0.3470 \end{pmatrix} \qquad H_{22} = \begin{pmatrix} 1.0033 & 0.2453 \\ 0.4945 & 0.4939 \end{pmatrix}$$
$$F_{11} = \begin{pmatrix} -0.4206 & -0.2806 \end{pmatrix} \qquad F_{12} = \begin{pmatrix} -0.6384 & -0.3228 \end{pmatrix}$$
$$F_{21} = \begin{pmatrix} -0.2083 & -0.1317 \end{pmatrix} \qquad F_{22} = \begin{pmatrix} -0.3105 & -0.0650 \end{pmatrix}$$

□

Example 3.4 Let us now revisit Example 3.1. Recall that we consider the two-rule TS model having the local matrices

$$A_1 = \begin{pmatrix} 2 & 0.04a+6.9 \\ -1 & 0.03b-2.9 \end{pmatrix} \quad B_1 = \begin{pmatrix} 0.03b-2.9 \\ 1 \end{pmatrix}$$
$$A_2 = \begin{pmatrix} 1 & 0.04a+6.9 \\ -1 & 0.03b-2.9 \end{pmatrix} \quad B_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

where a and b are real-valued parameters, $a, b \in [-4, 4]$. We have already compared the conditions in (Guerra and Vermeiren, 2004), and those of Theorem 3.1, which both involved 3 sums. Generalizing the conditions to 4 sums, we have $G_0^P = \{0, 0\}$ and $G_0^F = G_0^H = \{0\}$ for Theorem 3.3 and $G_0^P = \{-1, -1\}$ and $G_0^F = G_0^H = \{0, -1, -1\}$ for Theorem 3.4. The values of a and b for which we obtain solutions are presented in Figures 3.2(a) and 3.2(b). □



(a) Results using Theorem 3.3. (b) Results using Theorem 3.4.

Figure 3.2: Feasible solutions when using 4 sums for Example 3.1.

Let us now compare the conditions proposed in this thesis to that of Ding et al. (2006) and Lee et al. (2011). To consider the simplest case, the relaxation of Wang et al. (1996) is used on all the possible sums.

Example 3.5 Consider the two-rule TS model (Guerra and Vermeiren, 2004; Ding et al., 2006) having the local matrices

$$A_1 = \begin{pmatrix} 1 & -b \\ -1 & -0.5 \end{pmatrix} \quad B_1 = \begin{pmatrix} 5+b \\ 2b \end{pmatrix}$$
$$A_2 = \begin{pmatrix} 1 & b \\ -1 & -0.5 \end{pmatrix} \quad B_2 = \begin{pmatrix} 5-b \\ -2b \end{pmatrix}$$

where b is a real-valued parameter. The conditions of Theorem 3 in (Ding et al., 2006) are in fact a special case of Theorem 3.4 in Section 3.3.1, with $G_0^P = \{0,0\}$ and $G_0^F = G_0^H = \{0\}$. The number of sums is 4, and the maximum value of b for which the LMIs are feasible is $b = 1.547$. For comparison purposes, the conditions presented in (Guerra and Vermeiren, 2004) (special case of Theorem 3.3, with $G_0^P = G_0^F = G_0^H = \{0\}$) are feasible up to $b = 1.539$, although they only involve 3 sums. Applying Theorem 2 of Lee et al. (2011) with $G_0^P = \{-1,0,1\}$ and $G_0^H = G_0^F = \{-1,0\}$ using the relaxation of Wang et al. (1996) and without slack variables, $b = 1.565$ is obtained, but the conditions involve 5 sums. For the choice $G_0^P = \{0,1\}$ and $G_0^H = G_0^F = \{0\}$, which involves only 4 sums, the maximum b is $b = 1.54$.

Consider now the conditions presented in this chapter, in the light of the discussion in Section 3.3.2. For Theorem 3.3, to obtain only 4 sums, one can choose e.g., $G_0^P = \{0,0\}$ and $G_0^F = G_0^H = \{0\}$. With this choice, the maximum b obtained is $b = 1.547$, i.e., the same as in (Ding et al., 2006). For Theorem 3.4, according to the discussion in Section 3.3.2, we should choose $G_0^P = \{-1,-1\}$ and $G_0^F = G_0^H = \{0,-1,-1\}$. Indeed, with this we can obtain $b = 1.589$. Moreover, by choosing $G_0^P = \{-1\}$ and $G_0^F = G_0^H = \{0,-1\}$, which will lead to only 3 sums, we get $b = 1.553$, a better result than that obtained by the conditions of Ding et al. (2006). Consequently, a complexity reduction is also obtained. □

3.4 Extensions

In what follows, the results presented in Section 3.3.1 are extended to robust controllers, α-sample variation and H_∞ control.

3.4.1 Robust controllers

First, let us extend the conditions of Section 3.3.1 for the case when the system is described by

$$\boldsymbol{x}(k+1) = (\mathbb{A}_{G_0^A} + \Delta A)\boldsymbol{x}(k) + (\mathbb{B}_{G_0^B} + \Delta B)\boldsymbol{u}(k) \tag{3.15}$$

i.e., the local matrices are uncertain. The uncertainties considered are of the form $\Delta A = \mathbb{D}_{G_{0,a}^D}\Delta_a\mathbb{E}_{G_{0,a}^E}$ and $\Delta B = \mathbb{D}_{G_{0,b}^D}\Delta_b\mathbb{E}_{G_{0,b}^E}$, with $\Delta_a^T\Delta_a < I$, and $\Delta_b^T\Delta_b < I$.

The controller is of the form

$$\boldsymbol{u}(k) = -\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}\boldsymbol{x}(k)$$

and the closed-loop system can be expressed as

$$\boldsymbol{x}(k+1) = (\mathbb{A}_{G_0^A} + \Delta A - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} - \Delta B\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1})\boldsymbol{x}(k) \tag{3.16}$$

Then, using the two Lyapunov functions, the following results can be stated.

For Case 1, we have:

Corollary 3.5 *The closed-loop system* (3.16) *is asymptotically stable, if there exist* $P_{\boldsymbol{i}_j^P} = P_{\boldsymbol{i}_j^P}^T$, $\boldsymbol{i}_j^P = pr_{G_j^P}^{\boldsymbol{i}}$, *and* $H_{\boldsymbol{i}_j^H}$, $\boldsymbol{i}_j^H = pr_{G_j^H}^{\boldsymbol{i}}$, $\boldsymbol{i} \in \mathbb{I}_{G_V}$, $j = 0, 1$, $F_{\boldsymbol{i}_0^F}$, $\boldsymbol{i}_0^F = pr_{G_0^F}^{\boldsymbol{i}}$, $S_{\boldsymbol{i}_{0,a}^S} = S_{\boldsymbol{i}_{0,a}^S}^T > 0$, $\boldsymbol{i}_{0,a}^S = pr_{G_{0,a}^S}^{\boldsymbol{i}}$, *and* $S_{\boldsymbol{i}_{0,b}^S} = S_{\boldsymbol{i}_{0,b}^S}^T > 0$, $\boldsymbol{i}_0^{S_b} = pr_{G_{0,b}^S}^{\boldsymbol{i}}$, *where* $G_V = G_0^P \cup G_1^P \cup (G_0^F \oplus (G_0^B \cup G_{0,b}^E)) \cup (G_0^H \oplus (G_0^A \cup G_{0,a}^E)) \cup G_1^H \cup (G_{0,a}^S \oplus G_{0,a}^D \oplus G_{0,a}^D) \cup (G_{0,b}^S \oplus G_{0,b}^D \oplus G_{0,b}^D)$ *so that*

$$\begin{pmatrix} -\mathbb{P}_{G_0^P} & (*) & (*) & (*) \\ \mathbb{E}_{G_{0,a}^E}\mathbb{H}_{G_0^H} & -\mathbb{S}_{G_{0,a}^S} & (*) & (*) \\ \mathbb{E}_{G_{0,b}^E}\mathbb{F}_{G_0^F} & 0 & -\mathbb{S}_{G_{0,b}^S} & (*) \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & 0 & 0 & \begin{pmatrix} -\mathbb{H}_{G_1^H} - \mathbb{H}_{G_1^H}^T + \mathbb{P}_{G_1^P} \\ +\mathbb{D}_{G_{0,a}^D}\mathbb{S}_{G_{0,a}^S}\mathbb{D}_{G_{0,a}^D} + \mathbb{D}_{G_{0,b}^D}\mathbb{S}_{G_{0,b}^S}\mathbb{D}_{G_{0,b}^D} \end{pmatrix} \end{pmatrix} < 0 \tag{3.17}$$

Proof: Applying Theorem 3.3 for the system (3.16), we obtain

$$\begin{pmatrix} -\mathbb{P}_{G_0^P} & (*) \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} + \Delta A\mathbb{H}_{G_0^H} - \Delta B\mathbb{F}_{G_0^F} & -\mathbb{H}_{G_1^H} - \mathbb{H}_{G_1^H}^T + \mathbb{P}_{G_1^P} \end{pmatrix} < 0$$

Using the property that

$$\begin{aligned}\begin{pmatrix}0 & (*)\\ \Delta A\mathbb{H}_{G_0^H} & 0\end{pmatrix} &= \begin{pmatrix}0\\ \mathbb{D}_{G_{0,a}^D}\end{pmatrix}\Delta_a\left(\mathbb{E}_{G_{0,a}^E}\mathbb{H}_{G_0^H}\right)+(*)\\ &\leq \begin{pmatrix}0\\ \mathbb{D}_{G_{0,a}^D}\end{pmatrix}\mathbb{S}_{G_{0,a}^S}\begin{pmatrix}0\\ \mathbb{D}_{G_{0,a}^D}\end{pmatrix}^T+\left(\mathbb{E}_{G_{0,a}^E}\mathbb{H}_{G_0^H}\right)^T\mathbb{S}_{G_{0,a}^S}^{-1}\left(\mathbb{E}_{G_{0,a}^E}\mathbb{H}_{G_0^H}\right)\end{aligned}$$

for $\mathbb{S}_{G_{0,a}^S}=\mathbb{S}_{G_{0,a}^S}^T>0$, and similarly for

$$\begin{pmatrix}0 & (*)\\ \Delta B\mathbb{F}_{G_0^F} & 0\end{pmatrix}$$

and applying the Schur complement, we obtain directly (3.17). ■

For Case 2, we have:

Corollary 3.6 *The closed-loop system* (3.16) *is asymptotically stable, if there exist* $P_{\boldsymbol{i}_j^P}=P_{\boldsymbol{i}_j^P}^T$, $\boldsymbol{i}_j^P=pr^{\boldsymbol{i}}_{G_j^P}$, *and* $H_{\boldsymbol{i}_0^H}$, $\boldsymbol{i}_0^H=pr^{\boldsymbol{i}}_{G_0^H}$, $\boldsymbol{i}\in\mathbb{I}_{G_V}$, $j=0,1$, $F_{\boldsymbol{i}_0^F}$, $\boldsymbol{i}_0^F=pr^{\boldsymbol{i}}_{G_0^F}$, $S_{\boldsymbol{i}_{0,a}^S}=S_{\boldsymbol{i}_{0,a}^S}^T>0$, $\boldsymbol{i}_{0,a}^S=pr^{\boldsymbol{i}}_{G_{0,a}^S}$, *and* $S_{\boldsymbol{i}_{0,b}^S}=S_{\boldsymbol{i}_{0,b}^S}^T>0$, $\boldsymbol{i}_0^{S_b}=pr^{\boldsymbol{i}}_{G_{0,b}^S}$, *where* $G_V=G_0^P\cup G_1^P\cup(G_0^F\oplus(G_0^B\cup G_{0,b}^E))\cup(G_0^H\oplus(G_0^A\cup G_{0,a}^E))\cup(G_{0,a}^S\oplus G_{0,a}^D\oplus G_{0,a}^D)\cup(G_{0,b}^S\oplus G_{0,b}^D\oplus G_{0,b}^D)$ *so that*

$$\begin{pmatrix}-\mathbb{H}_{G_0^H}-\mathbb{H}_{G_0^H}^T+\mathbb{P}_{G_0^P} & (*) & (*) & (*)\\ \mathbb{E}_{G_{0,a}^E}\mathbb{H}_{G_0^H} & -\mathbb{S}_{G_{0,a}^S} & (*) & (*)\\ \mathbb{E}_{G_{0,b}^E}\mathbb{F}_{G_0^F} & 0 & -\mathbb{S}_{G_{0,b}^S} & (*)\\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H}-\mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & 0 & 0 & \begin{pmatrix}-\mathbb{P}_{G_1^P}+\mathbb{D}_{G_{0,a}^D}\mathbb{S}_{G_{0,a}^S}\mathbb{D}_{G_{0,a}^D}\\ +\mathbb{D}_{G_{0,b}^D}\mathbb{S}_{G_{0,b}^S}\mathbb{D}_{G_{0,b}^D}\end{pmatrix}\end{pmatrix}<0 \tag{3.18}$$

The proof follows the same lines as the proof of Corollary 3.5.

3.4.2 α-sample variation

In this section, we extend the results obtained in Section 3.3.1 using α-sample variation of the Lyapunov function (Kruszewski et al., 2008).

Recall that by using the controller (repeated here for convenience)

$$\boldsymbol{u}(k)=-\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}\boldsymbol{x}(k)$$

the closed-loop system is given by

$$\boldsymbol{x}(k+1)=\mathbb{A}_{G_0^A}\boldsymbol{x}(k)-\mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}\boldsymbol{x}(k)$$

For Case 1, the following result can be stated:

Theorem 3.7 *The closed-loop system* (3.10) *is asymptotically stable, if there exist* $P_{\boldsymbol{i}^P_j} = P^T_{\boldsymbol{i}^P_j}$, $\boldsymbol{i}^P_j = pr^{\boldsymbol{i}}_{G^P_j}$, $F_{\boldsymbol{i}^F_j}$, $\boldsymbol{i}^F_j = pr^{\boldsymbol{i}}_{G^F_j}$, *and* $H_{\boldsymbol{i}^H_j}$, $\boldsymbol{i}^H_j = pr^{\boldsymbol{i}}_{G^H_j}$, $\boldsymbol{i} \in \mathbb{I}_{G_V}$, $j = 0, 1, 2, \ldots, \alpha$, *where* $G_V = G^P_0 \cup G^P_\alpha \cup \bigcup_{i=0}^{\alpha-1}(G^F_i \oplus G^B_i) \cup \bigcup_{i=0}^{\alpha-1}(G^H_i \oplus G^A_i) \cup G^H_\alpha$ *so that*

$$\begin{pmatrix} -\mathbb{P}_{G^P_0} & (*) & \ldots & 0 \\ \mathbb{A}_{G^A_0}\mathbb{H}_{G^H_0} - \mathbb{B}_{G^B_0}\mathbb{F}_{G^F_0} & -\mathbb{H}_{G^H_1} - \mathbb{H}^T_{G^H_1} & \ldots & 0 \\ 0 & \mathbb{A}_{G^A_1}\mathbb{H}_{G^H_1} - \mathbb{B}_{G^B_1}\mathbb{F}_{G^F_1} & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \mathbb{P}_{G^P_\alpha} - \mathbb{H}^T_{G^H_\alpha} - \mathbb{H}_{G^H_\alpha} \end{pmatrix} < 0 \tag{3.19}$$

Remark: Note that again, G_V denotes the multiset of all the delays that appear in the sum in (3.19). The terms $\bigcup_{i=0}^{\alpha-1}(G^H_i \oplus G^A_i)$ and $\bigcup_{i=0}^{\alpha-1}(G^F_i \oplus G^B_i)$ are actually the delays that appear in the terms $\mathbb{A}_{G^A_i}\mathbb{H}_{G^H_i}$ and $\mathbb{B}_{G^B_i}\mathbb{F}_{G^F_i}$, $i = 1, 2, \ldots, \alpha - 1$.

Proof: Consider the Lyapunov function $V = \boldsymbol{x}(k)^T \mathbb{H}^{-T}_{G^H_0}\mathbb{P}_{G^P_0}\mathbb{H}^{-1}_{G^H_0}\boldsymbol{x}(k)$, with $P_{\boldsymbol{i}} = P^T_{\boldsymbol{i}} > 0$, for $\boldsymbol{i} \in \mathbb{I}_{G^P_0}$, $|G_0| = n_P$. The α-sample variation (see (Kruszewski et al., 2008)) of this Lyapunov function can be written as

$$\begin{aligned} \Delta V_\alpha =& V(k+\alpha) - V(k) = \\ =& \boldsymbol{x}(k+\alpha)^T \mathbb{H}^{-T}_{G^H_\alpha}\mathbb{P}_{G^P_\alpha}\mathbb{H}^{-1}_{G^H_\alpha}\boldsymbol{x}(k+\alpha) - \boldsymbol{x}(k)^T \mathbb{H}^{-T}_{G^H_0}\mathbb{P}_{G^P_0}\mathbb{H}^{-1}_{G^H_0}\boldsymbol{x}(k) \\ =& \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \\ \vdots \\ \boldsymbol{x}(k+\alpha) \end{pmatrix}^T \begin{pmatrix} -\mathbb{H}^{-T}_{G^H_0}\mathbb{P}_{G^P_0}\mathbb{H}^{-1}_{G^H_0} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \mathbb{H}^{-T}_{G^H_\alpha}\mathbb{P}_{G^P_\alpha}\mathbb{H}^{-1}_{G^H_0} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \\ \vdots \\ \boldsymbol{x}(k+\alpha) \end{pmatrix} \end{aligned}$$

The closed-loop system dynamics for α consecutive samples are

$$\begin{pmatrix} \Gamma_0 & -I & 0 & \ldots & 0 & 0 \\ 0 & \Gamma_1 & -I & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \Gamma_{\alpha-1} & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \\ \vdots \\ \boldsymbol{x}(k+\alpha) \end{pmatrix} = 0 \tag{3.20}$$

where $\Gamma_i = \mathbb{A}_{G^A_i} - \mathbb{B}_{G^B_i}\mathbb{F}_{G^F_i}(\mathbb{H}_{G^H_i})^{-1}$, $i = 0, 1, \ldots, \alpha - 1$. Using Lemma 2.13, $\Delta V_\alpha <$

0, if there exist $M \in \mathbb{R}^{n_x(\alpha+1)\times n_x\alpha}$ so that

$$\Omega = \begin{pmatrix} -\mathbb{H}_{G_0^H}^{-T}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \mathbb{H}_{G_\alpha^H}^{-T}\mathbb{P}_{G_\alpha^P}\mathbb{H}_{G_\alpha^H}^{-1} \end{pmatrix} + M \begin{pmatrix} \Gamma_0 & -I & 0 & \dots & 0 & 0 \\ 0 & \Gamma_1 & -I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \Gamma_{\alpha-1} & -I \end{pmatrix} + (*) < 0$$

To obtain LMI constraints that encompass classical cases, a choice is:

$$M = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \mathbb{H}_{G_1^H}^{-T} & 0 & 0 & \dots & 0 \\ 0 & \mathbb{H}_{G_2^H}^{-T} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \mathbb{H}_{G_\alpha^H}^{-T} \end{pmatrix}$$

With this choice of M, we have

$$\Omega = \begin{pmatrix} -\mathbb{H}_{G_0^H}^{-T}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1} & (*) & \dots & 0 \\ \begin{pmatrix} \mathbb{H}_{G_1^H}^{-T}\mathbb{A}_{G_0^A} \\ -\mathbb{H}_{G_1^H}^{-T}\mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} \end{pmatrix} & -\mathbb{H}_{G_1^H}^{-T} - \mathbb{H}_{G_1^H}^{-1} & \dots & 0 \\ 0 & \begin{pmatrix} \mathbb{H}_{G_2^H}^{-T}\mathbb{A}_{G_1^A} \\ -\mathbb{H}_{G_2^H}^{-T}\mathbb{B}_{G_1^B}\mathbb{F}_{G_1^F}\mathbb{H}_{G_1^H}^{-1} \end{pmatrix} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \begin{pmatrix} \mathbb{H}_{G_\alpha^H}^{-T}\mathbb{P}_{G_\alpha^P}\mathbb{H}_{G_\alpha^H}^{-1} \\ -\mathbb{H}_{G_\alpha^H}^{-T} - \mathbb{H}_{G_\alpha^H}^{-1} \end{pmatrix} \end{pmatrix}$$

Using Property 1 with $\mathrm{diag}(\mathbb{H}_{G_i^H}^T)$, $i = 1, 2, \dots, \alpha$, $\Omega < 0$ gives directly (3.19). ■

For Case 2, we have the following conditions:

Theorem 3.8 *The closed-loop system* (3.10) *is asymptotically stable, if there exist* $P_{\boldsymbol{i}_j^P} = P_{\boldsymbol{i}_j^P}^T$, $\boldsymbol{i}_j^P = pr^{\boldsymbol{i}}_{G_j^P}$, $F_{\boldsymbol{i}_j^F}$, $\boldsymbol{i}_j^F = pr^{\boldsymbol{i}}_{G_j^F}$, *and* $H_{\boldsymbol{i}_j^H}$, $\boldsymbol{i}_j^H = pr^{\boldsymbol{i}}_{G_j^H}$, $\boldsymbol{i} \in \mathbb{I}_{G_V}$, $j = 0, 1, 2, \dots, \alpha$, *where* $G_V = G_0^P \cup G_\alpha^P \cup \bigcup_{i=0}^{\alpha-1}(G_i^F \oplus G_i^B) \cup \bigcup_{i=0}^{\alpha-1}(G_i^H \oplus G_i^A) \cup G_\alpha^H$ *so that*

$$\begin{pmatrix} \begin{pmatrix} -\mathbb{H}_{G_0^H} - \mathbb{H}_{G_0^H}^T \\ +\mathbb{P}_{G_0^P} \end{pmatrix} & (*) & \dots & 0 & 0 \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & -\mathbb{H}_{G_1^H} - \mathbb{H}_{G_1^H}^T & \dots & 0 & 0 \\ 0 & \mathbb{A}_{G_1^A}\mathbb{H}_{G_1^H} - \mathbb{B}_{G_1^B}\mathbb{F}_{G_1^F} & \dots & 0 & \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & -\mathbb{H}_{G_{\alpha-1}^H} - \mathbb{H}_{G_{\alpha-1}^H}^T & 0 \\ 0 & 0 & \dots & \begin{pmatrix} \mathbb{A}_{G_{\alpha-1}^A}\mathbb{H}_{G_{\alpha-1}^H} \\ -\mathbb{B}_{G_{\alpha-1}^B}\mathbb{F}_{G_{\alpha-1}^F} \end{pmatrix} & -\mathbb{P}_{G_\alpha^P} \end{pmatrix} < 0 \tag{3.21}$$

Proof: Consider the Lyapunov function $V = \boldsymbol{x}(k)^T \mathbb{P}_{G_0^P}^{-1} \boldsymbol{x}(k)$, with $P_{\boldsymbol{i}} = P_{\boldsymbol{i}}^T > 0$, for $\boldsymbol{i} \in \mathbb{I}_{G_0^P}$, $|G_0| = n_P$. The α-sample variation is

$$\begin{aligned} \Delta V_\alpha =& V(k+\alpha) - V(k) = \\ =& \boldsymbol{x}(k+\alpha)^T \mathbb{P}_{G_\alpha^P}^{-1} \boldsymbol{x}(k+\alpha) - \boldsymbol{x}(k)^T \mathbb{P}_{G_0^P}^{-1} \boldsymbol{x}(k) \\ =& \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \\ \vdots \\ \boldsymbol{x}(k+\alpha) \end{pmatrix}^T \begin{pmatrix} -\mathbb{P}_{G_0^P}^{-1} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \mathbb{P}_{G_\alpha^P}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \\ \vdots \\ \boldsymbol{x}(k+\alpha) \end{pmatrix} \end{aligned}$$

together with the equality constraint (3.20).

To use Lemma 2.13, one can choose for instance

$$M = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \mathbb{H}_{G_1^H}^{-T} & 0 & \dots & 0 & 0 \\ 0 & \mathbb{H}_{G_2^H}^{-T} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \mathbb{H}_{G_{\alpha-1}^H}^{-T} & 0 \\ 0 & 0 & \dots & 0 & \mathbb{P}_{G_\alpha^P}^{-1} \end{pmatrix}$$

Congruence with the matrix $\mathrm{diag}(\mathbb{H}_{G_i^H}, \mathbb{P}_{G_\alpha^P})$, $i = 1, 2, \dots, \alpha - 1$ and applying Property 5 leads directly to (3.21). ■

Remark: Similarly to Section 3.3.1, depending on the exact sets of indices used, G_0^P, G_0^F, and G_0^H, relaxations such as (Wang et al., 1996; Tanaka et al., 1998; Tuan et al., 2001; Sala and Ariño, 2007) can be used.

3.4.3 H_∞-control

In this section, we consider H_∞ control using the controller (3.10). Consider then the system expressed as:

$$\begin{aligned} \boldsymbol{x}(k+1) &= \mathbb{A}_{G_0^A}\boldsymbol{x}(k) + \mathbb{B}_{G_0^B}\boldsymbol{u}(k) + \mathbb{E}_{G_0^E}\boldsymbol{w}(k) \\ \boldsymbol{y}(k) &= \mathbb{C}_{G_0^C}\boldsymbol{x}(k) + \mathbb{D}_{G_0^D}\boldsymbol{u}(k) + \mathbb{K}_{G_0^K}\boldsymbol{w}(k) \end{aligned} \tag{3.22}$$

where $\boldsymbol{w}(k)$ denotes the disturbance affecting the system.

The closed-loop system is given by

$$\begin{aligned} \boldsymbol{x}(k+1) &= \mathbb{A}_{G_0^A}\boldsymbol{x}(k) - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}^{-1}_{G_0^H}\boldsymbol{x}(k) + \mathbb{E}_{G_0^E}\boldsymbol{w}(k) \\ \boldsymbol{y}(k) &= \mathbb{C}_{G_0^C}\boldsymbol{x}(k) - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F}\mathbb{H}^{-1}_{G_0^H}\boldsymbol{x}(k) + \mathbb{K}_{G_0^K}\boldsymbol{w}(k) \end{aligned} \tag{3.23}$$

For Case 1, the following result can be stated:

Theorem 3.9 *The closed-loop system* (3.23) *is asymptotically stable, and the attenuation is at least γ if there exist $\gamma > 0$, $P_{\boldsymbol{i}_j^P} = P^T_{\boldsymbol{i}_j^P}$, $\boldsymbol{i}_j^P = pr^{\boldsymbol{i}}_{G_j^P}$, $j = 0, 1$, $F_{\boldsymbol{i}_0^F}$, $\boldsymbol{i}_0^F = pr^{\boldsymbol{i}}_{G_0^F}$, and $H_{\boldsymbol{i}_0^H}$, $\boldsymbol{i}_0^H = pr^{\boldsymbol{i}}_{G_0^H}$, $\boldsymbol{i} \in \mathbb{I}_{G_V}$, where $G_V = G_0^P \cup G_1^P \cup (G_0^F \oplus G_0^B) \cup (G_0^H \oplus G_0^A) \cup G_0^K \cup G_0^E$ so that*

$$\begin{pmatrix} -\mathbb{P}_{G_0^P} & (*) & (*) & (*) \\ 0 & -\gamma I & (*) & (*) \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & \mathbb{E}_{G_0^E} & -\mathbb{H}_{G_1^H} - \mathbb{H}^T_{G_1^H} + \mathbb{P}_{G_1^P} & (*) \\ \mathbb{C}_{G_0^C}\mathbb{H}_{G_0^H} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F} & \mathbb{K}_{G_0^K} & 0 & -\gamma I \end{pmatrix} < 0 \tag{3.24}$$

Proof: Consider the Lyapunov function $V = \boldsymbol{x}(k)^T\mathbb{H}^{-T}_{G_0^H}\mathbb{P}_{G_0^P}\mathbb{H}^{-1}_{G_0^H}\boldsymbol{x}(k)$, with $P_{\boldsymbol{i}} = P_{\boldsymbol{i}}^T > 0$, for $\boldsymbol{i} \in \mathbb{I}_{G_0^P}$, $|G_0^P| = n_P$. The difference is

$$\begin{aligned} \Delta V_1 =& V(k+1) - V(k) = \\ =& \boldsymbol{x}(k+1)^T\mathbb{P}^{-1}_{G_1^P}\boldsymbol{x}(k+1) - \boldsymbol{x}(k)^T\mathbb{P}^{-1}_{G_0^P}\boldsymbol{x}(k) \\ =& \begin{pmatrix}\boldsymbol{x}(k)\\ \boldsymbol{w}(k)\end{pmatrix}^T \begin{pmatrix} \Omega - \mathbb{H}^{-T}_{G_0^H}\mathbb{P}_{G_0^P}\mathbb{H}^{-1}_{G_0^H} & (*) \\ \mathbb{E}^T_{G_0^E}\mathbb{H}^{-T}_{G_1^H}\mathbb{P}_{G_1^P}\mathbb{H}^{-1}_{G_1^H}(\mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}^{-1}_{G_0^H}) & \mathbb{E}^T_{G_0^E}\mathbb{H}^{-T}_{G_1^H}\mathbb{P}_{G_1^P}\mathbb{H}^{-1}_{G_1^H}\mathbb{E}_{G_0^E} \end{pmatrix} \begin{pmatrix}\boldsymbol{x}(k)\\ \boldsymbol{w}(k)\end{pmatrix} \end{aligned}$$

where

$$\Omega = (\mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}^{-1}_{G_0^H})^T\mathbb{H}^{-T}_{G_1^H}\mathbb{P}_{G_1^P}\mathbb{H}^{-1}_{G_1^H}(\mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}^{-1}_{G_0^H})$$

Since $\boldsymbol{y}(k)^T\boldsymbol{y}(k) - \gamma^2\boldsymbol{w}(k)^T\boldsymbol{w}(k) < 0$, results that

$$\begin{pmatrix}\boldsymbol{x}(k)\\ \boldsymbol{w}(k)\end{pmatrix}^T \begin{pmatrix} (\mathbb{C}_{G_0^C} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F}\mathbb{H}^{-1}_{G_0^H})^T(\mathbb{C}_{G_0^C} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F}\mathbb{H}^{-1}_{G_0^H}) & (*) \\ \mathbb{K}^T_{G_0^K}(\mathbb{C}_{G_0^C} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F}\mathbb{H}^{-1}_{G_0^H}) & \mathbb{K}^T_{G_0^K}\mathbb{K}_{G_0^K} - \gamma^2 I \end{pmatrix} \begin{pmatrix}\boldsymbol{x}(k)\\ \boldsymbol{w}(k)\end{pmatrix} < 0 \tag{3.25}$$

With the S-procedure, $\Delta V_1 < 0$ under constraint (3.25), if

$$\begin{pmatrix} \Omega - \mathbb{H}_{G_0^H}^{-T}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1} & (*) \\ \mathbb{E}_{G_0^E}^T\mathbb{H}_{G_1^H}^{-T}\mathbb{P}_{G_1^P}\mathbb{H}_{G_1^H}^{-1}(\mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}) & \mathbb{E}_{G_0^E}^T\mathbb{H}_{G_1^H}^{-T}\mathbb{P}_{G_1^P}\mathbb{H}_{G_1^H}^{-1}\mathbb{E}_{G_0^E} \end{pmatrix}$$
$$+\begin{pmatrix} \gamma^{-1}(\mathbb{C}_{G_0^C} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1})^T(\mathbb{C}_{G_0^C}\mathbb{H}_{G_0^H} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}) & (*) \\ \gamma^{-1}\mathbb{K}_{G_0^K}^T(\mathbb{C}_{G_0^C} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}) & \gamma^{-1}\mathbb{K}_{G_0^K}^T\mathbb{K}_{G_0^K} - \gamma I \end{pmatrix}$$
$$= \begin{pmatrix} (\mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1})^T \\ \mathbb{E}_{G_0^E}^T \end{pmatrix} \mathbb{H}_{G_1^H}^{-T}\mathbb{P}_{G_1^P}\mathbb{H}_{G_1^H}^{-1} \begin{pmatrix} \mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} & \mathbb{E}_{G_0^E} \end{pmatrix}$$
$$+\gamma^{-1}\begin{pmatrix} (\mathbb{C}_{G_0^C} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1})^T \\ \mathbb{K}_{G_0^K}^T \end{pmatrix}\begin{pmatrix} \mathbb{C}_{G_0^C} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} & \mathbb{K}_{G_0^K} \end{pmatrix}$$
$$-\begin{pmatrix} \mathbb{H}_{G_0^H}^{-T}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1} & 0 \\ 0 & \gamma I \end{pmatrix}$$
$$= (*)\begin{pmatrix} \mathbb{H}_{G_1^H}^{-T}\mathbb{P}_{G_1^P}\mathbb{H}_{G_1^H}^{-1} & 0 \\ 0 & \gamma^{-1}I \end{pmatrix}\begin{pmatrix} \mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} & \mathbb{E}_{G_0^E} \\ \mathbb{C}_{G_0^C} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} & \mathbb{K}_{G_0^K} \end{pmatrix}$$
$$-\begin{pmatrix} \mathbb{H}_{G_0^H}^{-T}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1} & 0 \\ 0 & \gamma I \end{pmatrix}$$

Congruence with $\begin{pmatrix} \mathbb{H}_{G_0^H}^T & 0 \\ 0 & I \end{pmatrix}$ gives

$$(*)\begin{pmatrix} \mathbb{H}_{G_1^H}^{-T}\mathbb{P}_{G_1^P}\mathbb{H}_{G_1^H}^{-1} & 0 \\ 0 & \gamma^{-1}I \end{pmatrix}\begin{pmatrix} \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & \mathbb{E}_{G_0^E} \\ \mathbb{C}_{G_0^C}\mathbb{H}_{G_0^H} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F} & \mathbb{K}_{G_0^K} \end{pmatrix} - \begin{pmatrix} \mathbb{P}_{G_0^P} & 0 \\ 0 & \gamma I \end{pmatrix} < 0$$

Using the Schur complement and Property 5, (3.24) results directly. ■

For Case 2, we have

Theorem 3.10 *The closed-loop system* (3.23) *is asymptotically stable, and the attenuation is at least* γ *if there exist* $\gamma > 0$, $P_{\boldsymbol{i}_j^P} = P_{\boldsymbol{i}_j^P}^T$, $\boldsymbol{i}_j^P = pr_{G_j^P}^{\boldsymbol{i}}$, $j = 0, 1$, $F_{\boldsymbol{i}_0^F}$, $\boldsymbol{i}_0^F = pr_{G_0^F}^{\boldsymbol{i}}$, *and* $H_{\boldsymbol{i}_0^H}$, $\boldsymbol{i}_0^H = pr_{G_0^H}^{\boldsymbol{i}}$, $\boldsymbol{i} \in \mathbb{I}_{G_V}$, *where* $G_V = G_0^P \cup G_1^P \cup (G_0^F \oplus G_0^B) \cup (G_0^H \oplus G_0^A) \cup G_0^K \cup G_0^E$ *so that*

$$\begin{pmatrix} -\mathbb{H}_{G_0^H} - \mathbb{H}_{G_0^H}^T + \mathbb{P}_{G_0^P} & (*) & (*) & (*) \\ 0 & -\gamma I & (*) & (*) \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & \mathbb{E}_{G_0^E} & -\mathbb{P}_{G_1^P} & (*) \\ \mathbb{C}_{G_0^C}\mathbb{H}_{G_0^H} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F} & \mathbb{K}_{G_0^K} & 0 & -\gamma I \end{pmatrix} < 0 \tag{3.26}$$

Proof: Consider now the Lyapunov function $V = \boldsymbol{x}(k)^T \mathbb{P}_{G_0^P}^{-1} \boldsymbol{x}(k)$, with $P_{\boldsymbol{i}} = P_{\boldsymbol{i}}^T > 0$, for $\boldsymbol{i} \in \mathbb{I}_{G_0^P}$, $|G_0^P| = n_P$. The difference is

$$
\begin{aligned}
\Delta V_1 =& V(k+1) - V(k) = \\
=& \boldsymbol{x}(k+1)^T \mathbb{P}_{G_1^P}^{-1} \boldsymbol{x}(k+1) - \boldsymbol{x}(k)^T \mathbb{P}_{G_0^P}^{-1} \boldsymbol{x}(k) \\
=& \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{w}(k) \end{pmatrix}^T \begin{pmatrix} (\mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1})^T \mathbb{P}_{G_1^P}^{-1} (\mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}) - \mathbb{P}_{G_0^P}^{-1} & (*) \\ \mathbb{E}_{G_0^E}^T \mathbb{P}_{G_1^P}^{-1} (\mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}) & \mathbb{E}_{G_0^E}^T \mathbb{P}_{G_1^P}^{-1} \mathbb{E}_{G_0^E} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{w}(k) \end{pmatrix}
\end{aligned}
$$

Since $\boldsymbol{y}(k)^T \boldsymbol{y}(k) - \gamma^2 \boldsymbol{w}(k)^T \boldsymbol{w}(k) < 0$, results that

$$
\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{w}(k) \end{pmatrix}^T \begin{pmatrix} (\mathbb{C}_{G_0^C} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1})^T (\mathbb{C}_{G_0^C} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}) & (*) \\ \mathbb{K}_{G_0^K}^T (\mathbb{C}_{G_0^C} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}) & \mathbb{K}_{G_0^K}^T \mathbb{K}_{G_0^K} - \gamma^2 I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{w}(k) \end{pmatrix} < 0
$$

With the S-procedure,

$$
\begin{aligned}
& \begin{pmatrix} (\mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1})^T \mathbb{P}_{G_1^P}^{-1} (\mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}) - \mathbb{P}_{G_0^P}^{-1} & (*) \\ \mathbb{E}_{G_0^E}^T \mathbb{P}_{G_1^P}^{-1} (\mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}) & \mathbb{E}_{G_0^E}^T \mathbb{P}_{G_1^P}^{-1} \mathbb{E}_{G_0^E} \end{pmatrix} \\
&+ \begin{pmatrix} \gamma^{-1} (\mathbb{C}_{G_0^C} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1})^T (\mathbb{C}_{G_0^C}\mathbb{H}_{G_0^H} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}) & (*) \\ \gamma^{-1} \mathbb{K}_{G_0^K}^T (\mathbb{C}_{G_0^C} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}) & \gamma^{-1} \mathbb{K}_{G_0^K}^T \mathbb{K}_{G_0^K} - \gamma I \end{pmatrix} \\
&= (*) \begin{pmatrix} \mathbb{P}_{G_1^P}^{-1} & 0 \\ 0 & \gamma^{-1} I \end{pmatrix} \begin{pmatrix} \mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} & \mathbb{E}_{G_0^E} \\ \mathbb{C}_{G_0^C} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} & \mathbb{K}_{G_0^K} \end{pmatrix} - \begin{pmatrix} \mathbb{P}_{G_0^P}^{-1} & 0 \\ 0 & \gamma I \end{pmatrix}
\end{aligned}
$$

Congruence with $\begin{pmatrix} \mathbb{H}_{G_0^H}^T & 0 \\ 0 & I \end{pmatrix}$ gives

$$
(*) \begin{pmatrix} \mathbb{P}_{G_1^P}^{-1} & 0 \\ 0 & \gamma^{-1} I \end{pmatrix} \begin{pmatrix} \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & \mathbb{E}_{G_0^E} \\ \mathbb{C}_{G_0^C}\mathbb{H}_{G_0^H} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F} & \mathbb{K}_{G_0^K} \end{pmatrix} - \begin{pmatrix} \mathbb{H}_{G_0^H}^T \mathbb{P}_{G_0^P}^{-1} \mathbb{H}_{G_0^H} & 0 \\ 0 & \gamma I \end{pmatrix} < 0
$$

Using the Schur complement and Property 5, (3.26) results directly. ■

3.5 Summary

This chapter presented a general framework for the design of nonquadratic controllers for TS fuzzy models. Two methods have been described, the difference between them coming from the structure of the Lyapunov function used. It has been shown that the proposed controllers include controllers reported in the recent literature. The design methods have been illustrated on numerical examples and extended to robust control, α-sample variation, and H_∞ control.

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Part II

Periodic and switching systems

Chapter 4

Introduction and outline

4.1 Introduction

This part of the thesis deals with the stability analysis, controller and observer design for switching and periodic systems described by TS models.

First, we consider a particular class of nonlinear models with periodic parameters which can be represented by periodic TS models. Periodic models can be found in numerous domains such as automotive, aeronautic, aerospace and even computer control of industrial process. For example in (Chauvin et al., 2005), a periodic dynamic model is used to estimate the air/fuel ratio in each cylinder on an internal combustion engine, Gaiani et al. (2004) proposed a periodic model for the rotor blades of helicopter, Theron et al. (2007) deal with the problem of on-board automatic station keeping of a small spacecraft on a specific orbit of reference and proposes a periodic state feedback control law. Other examples are provided by Bittanti and Colaneri (2000) related to computer control and communication systems.

The stability of linear periodic systems is characterized by the monodromy transition matrix and by its eigenvalues, called the characteristic multipliers (often referred to the poles of the system). If all of the characteristic multipliers are in the open unit disc of the complex plane then the system is asymptotically stable (Farkas, 1994). Concerning the stabilization problem of those models, results are available in (Farges et al., 2007). For models including time varying delays, Stepan and Insperger (2006) proposed methods based on Floquet's transformation, which is only applied to autonomous systems, and led to conditions for exponential stability.

Some extensions exist to polytopic LPV periodic models, where the stability analysis is based on the use of quadratic (Farges et al., 2007; Arzelier et al., 2005) or non-quadratic (Daafouz et al., 2002) Lyapunov functions. In the nonlinear TS context, (Kruszewski and Guerra, 2007; Kerkeni et al., 2011) are dealing with stabilization of discrete TS models with periodic parameters.

We consider stability analysis and controller design for periodic, discrete-time

TS models. To derive the conditions, we use a periodic non-quadratic Lyapunov function. Although in general, stability analysis and controller design for TS models relies on the stability and controllability, respectively, of each local model, using this Lyapunov function we are able to prove the stability of a periodic TS system having non-stable local models and even unstable subsystems. Moreover, with the conditions derived using this Lyapunov function we are able to design stabilizing control laws for switching systems in the case when the local models of the subsystems are not stable and not stabilizable.

Second, we consider switching systems, which are a class of hybrid systems that switch between a family of modes or subsystems. Such models can be found in numerous and various domains (Zwart et al., 2010; Venkataramanan et al., 2002; Pasamontes et al., 2011; Widyotriatmo and Hong, 2012; Moustris and Tzafestas, 2011; Zhao and Spong, 2001; Li et al., 2012, 2014), such as automotive, network controlled application, DC converters, mobile robots, etc. In the case of automotive applications, switching approaches have been used for different parts of the vehicle: engine control, HCCI combustion (Liao et al., 2013), air path with turbocharger (Nguyen et al., 2012b,a, 2013), clutch actuator control (Langjord et al., 2008). Multi-cellular converters are components which require to control several switches with high frequency to lead to a desired level of conversion. A way to consider the different possible modes is to use a switching structure (Hauroigné et al., 2012).

In the previously listed applications, the problems addressed are the analysis of stability and/or the development of controllers or observers. Originally, mainly continuous time switching systems were considered but recently discrete time approach have been developed (Chen et al., 2012; Duan and Wu, 2012; Hetel et al., 2011).

Linear switching systems where the switching laws can be arbitrarily chosen have been considered in (Altafini, 2002) to study the reachable set of such systems. Stabilization and tracking conditions for continuous-time linear switching systems have been developed in (Baglietto et al., 2013; Battistelli, 2013), delay-dependent stabilization in (Kim et al., 2008), and observability with unknown input has been investigated in (Boukhobza and Hamelin, 2011). The results for switching systems have been applied in (Yongqiang Guan and et al., 2013) for the decentralized stabilization of multiagent systems. State-feedback controller design for nonlinear switching systems has been presented in (Blanchini et al., 2007) and optimal control in (Bengea and DeCarlo, 2005). A notable result, although for continuous-time linear switching systems is the one in (Ji et al., 2007), which concerns the design of switching sequences for stabilization and proves that it is sufficient for stabilization to employ a periodic switching law.

It is well-known that by switching between two – independently stable – subsystems the switching system can be destabilized and conversely, by switching between unstable subsystems, the states can be made to converge to zero. Our goal is to design the switching law that stabilizes a given switching system. Switched TS systems have been investigated mainly in the continuous case where the stability is based on the use of a quadratic Lyapunov function (Tanaka et al., 2001; Lam et al., 2002, 2004; Ohtake et al., 2006) or a piecewise one (Feng, 2003, 2004). For discrete-time switching TS models, few results exist (Doo et al., 2003; Dong and Yang, 2009). We consider arbitrary switching discrete-time systems and investigate under which conditions they can be stabilized by a suitably chosen switching law. Instead of attempting to stabilize each subsystem so that the overall system to be stable, we stabilize the system by switching between possible unstable subsystems. We first present sufficient conditions for being able to stabilize a discrete-time switching nonlinear system simply by switching between the subsystems; and then construct a switching law that stabilizes the system.

The material in this part is based on the following publications:

(P3) Zs. Lendek, J. Lauber, and T. M. Guerra, Periodic Lyapunov functions for periodic TS systems. Systems & Control Letters, vol. 62 , no. 4, pages 303-310, 2013.

(P4) Zs. Lendek, P. Raica, J. Lauber, T. M. Guerra, Finding a stabilizing switching law for switching TS models. International Journal of Systems Science, vol. 47, pages 2762-2772, 2016.

(P5) Zs. Lendek, P. Raica, J. Lauber, T. M. Guerra, Observer Design for Discrete-Time Switching Nonlinear Models. In Hybrid Dynamical Systems, series Lecture Notes in Control and Information Sciences, M. Djemai, M. Defoort, Editors, vol. 457, pages 27-58. Springer International Publishing, 2015.

(P6) Zs. Lendek, P. Raica, J. Lauber, T. M. Guerra, Nonquadratic stabilization of switching TS systems. In Preprints of the 2014 IFAC World Congress, pages 7970-7975, Cape Town, South Africa, August 2014.

(P7) Zs. Lendek, P. Raica, J. Lauber, T. M. Guerra, Observer design for switching nonlinear systems. In Proceedings of the 2014 IEEE World Congress on Computational Intelligence, IEEE International Conference on Fuzzy Systems, pages 1-6, Beijing, China, July 2014.

(P8) Zs. Lendek, J. Lauber, T. M. Guerra, P. Raica, Stability analysis of switching TS models using alpha-samples approach. In Proceedings of the 2013 IFAC International Conference on Intelligent Control and Automation Science, pages 207-211, Chengdu, China, September 2013.

(P9) Zs. Lendek, J. Lauber, T. M. Guerra, P. Raica, On stabilization of discrete-time periodic TS systems. In Proceedings of the 2013 IEEE International Conference on Fuzzy Systems, pages 1-7, Hyderabad, India, July 2013.

(P10) Zs. Lendek, J. Lauber, T.M. Guerra, Switching fuzzy observers for periodic TS systems. In Proceedings of the 2012 IEEE International Conference on

Automation, Quality and Testing, Robotics, pages 1-6, Cluj, Romania, May 2012.

(P11) Zs. Lendek, J. Lauber, T.M. Guerra, Switching Lyapunov functions for periodic TS systems. In Proceedings of the 1st IFAC Conference on Embedded Systems, Computational Intelligence and Telematics in Control, pages 1-6, Wurzburg, Germany, April 2012.

4.2 Motivating examples

In the literature, one of the main assumptions on switching systems is that the switching can occur at any time, between any two subsystems. However, for periodic systems, the extra knowledge of when and between which subsystems the switching will occur can lead to more relaxed conditions. Consider for instance, the switching system composed of two linear subsystems, with state matrices

$$A_1 = \begin{pmatrix} 0.7 & 0.8 \\ 0.2 & 0.3 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0.5 & 0.8 \\ 0.2 & a \end{pmatrix}$$

where a is a real-valued parameter. Using e.g., the results in (Daafouz et al., 2002), one is able to prove stability of the switching system for $a \in [-0.89, 0.67]$. However, if we know that the system switches from one subsystem to the other at every time instant, the stability of the switching system can be proven for $a \in [-5.1, 1.1]$. Consequently, by using the knowledge of when and how a periodic system switches, one can significantly relax the stability conditions. In what follows, we investigate this possibility for TS systems. We also present extensions for controller and observer design.

To show the interest in switching systems, consider a system with three subsystems as follows:

$$\Sigma_1 : \boldsymbol{x}(k+1) = h_{1,1}(\boldsymbol{x}(k))A_{1,1}\boldsymbol{x}(k) + h_{1,2}(\boldsymbol{x}(k))A_{1,2}\boldsymbol{x}(k)$$

$$A_{1,1} = \begin{pmatrix} -0.44 & -0.26 \\ -0.65 & 0.62 \end{pmatrix} \qquad A_{1,2} = \begin{pmatrix} 1.1 & -0.2 \\ 0.53 & -0.27 \end{pmatrix}$$

$$h_{1,1} = \frac{1-\sin(x_1(k))}{2} \qquad h_{1,2} = 1 - h_{1,1}$$

$$\Sigma_2 : \boldsymbol{x}(k+1) = h_{2,1}(\boldsymbol{x}(k))A_{2,1}\boldsymbol{x}(k) + h_{2,2}(\boldsymbol{x}(k))A_{2,2}\boldsymbol{x}(k)$$

$$A_{2,1} = \begin{pmatrix} 0.02 & 0.6 \\ -0.22 & -0.44 \end{pmatrix} \qquad A_{2,2} = \begin{pmatrix} 0.32 & -0.15 \\ -1 & 0.8 \end{pmatrix}$$

$$h_{2,1} = \frac{1-\cos(x_1(k))}{2} \qquad h_{2,2} = 1 - h_{2,1}$$

$$\Sigma_3 : \boldsymbol{x}(k+1) = h_{3,1}(\boldsymbol{x}(k))A_{3,1}\boldsymbol{x}(k) + h_{3,2}(\boldsymbol{x}(k))A_{3,2}\boldsymbol{x}(k)$$

$$A_{3,1} = \begin{pmatrix} 2 & 0.5 \\ 0.5 & 2 \end{pmatrix} \qquad A_{3,2} = \begin{pmatrix} 1 & 0.1 \\ 0.5 & 3 \end{pmatrix}$$

$$h_{3,1} = \frac{1-\exp(-x_1^2(k))}{2} \qquad h_{3,2} = 1 - h_{3,1}$$

where $x_1(k)$ denotes the first element of the state vector $\boldsymbol{x}$ at time k. One can switch from each subsystem to any other one and any subsystem can be active for any number of samples. However, $A_{1,2}$ and $A_{2,2}$ are not Schur and both $A_{3,1}$ and $A_{3,2}$ have eigenvalues larger than 1. Our goal is to find a switching law that stabilizes the switching system above.

Since none of the subsystems is stable, no existing result in the literature can prove the stability of this system. Moreover, just switching to one subsystem and keeping it continuously active is not a solution. However, by switching continuously between the first and second subsystem, the states converge to zero. This can be proven by using a periodic Lyapunov function, such as the one proposed in (Lendek et al., 2013). Consequently, in order to stabilize the system, when starting from the third subsystem, one can switch to the first or second one and then switch between these two.

4.3 Notation

In this part we consider stability analysis and controller and observer design of discrete-time periodic and switching TS systems. For stability analysis, we consider subsystems of the form

$$\begin{aligned} \boldsymbol{x}(k+1) &= \sum_{i=1}^{r_j} h_{ji}(\boldsymbol{z}_j(k))A_{j,i}\boldsymbol{x}(k) \\ &= A_{j,z}\boldsymbol{x}(k) \end{aligned} \tag{4.1}$$

for controller design

$$\boldsymbol{x}(k+1) = A_{j,z}\boldsymbol{x}(k) + B_{j,z}\boldsymbol{u}(k) \tag{4.2}$$

and for observer design

$$\begin{aligned} \boldsymbol{x}(k+1) &= A_{j,z}\boldsymbol{x}(k) + B_{j,z}\boldsymbol{u}(k) \\ \boldsymbol{y}(k) &= C_{j,z}\boldsymbol{x}(k) \end{aligned} \tag{4.3}$$

where j is the number of the current subsystem, $j = 1, 2, \ldots, n_s$, n_s being the number of the subsystems, $\boldsymbol{x}$ denotes the state vector, $\boldsymbol{u}$ is the input, $\boldsymbol{y}$ is the output, r_j is the number of local models in the jth subsystem, $\boldsymbol{z}_j$ is the scheduling vector, h_{ji}, $i = 1, 2, \ldots, r_j$ are normalized membership functions, and $A_{j,i}$, $B_{j,i}$, $C_{j,i}$, $i = 1, 2, \ldots, r_j$, $j = 1, 2, \ldots, n_s$, are the local models.

For periodic systems, the subsystems defined above are activated in a sequence $\underbrace{1,1,\ldots,1}_{p_1},\underbrace{2,2,\ldots,2}_{p_2},\ldots,\underbrace{n_s,n_s,\ldots,n_s}_{p_{n_s}},\underbrace{1,1,\ldots,1}_{p_1}$, etc., where p_i denotes the number of samples for which the ith subsystem is active. In what follows, we will refer to p_i as the period of the ith subsystem.

An underlined variable $\underline{j}$ denotes the modulo of the variable, i.e., $\underline{j} = (j \mod n_s) + 1$.

We use a Lyapunov function defined only in the switching instants. This also means that the α-difference in the Lyapunov function corresponds to α consecutive switches in the system. To illustrate this, consider consider a switching TS model consisting of two subsystems, each with period 2, i.e., we have:

$$\boldsymbol{x}(k+1) = \begin{cases} \sum_{i=1}^{r_1} h_{1i}(\boldsymbol{z}_1(k))A_{1i}\boldsymbol{x}(k) & \text{if } k = 4m, 4m+1 \\ \sum_{i=1}^{r_2} h_{2i}(\boldsymbol{z}_2(k))A_{2i}\boldsymbol{x}(k) & \text{if } k = 4m+2, 4m+3 \end{cases} \tag{4.4}$$

The switching in the system and in the Lyapunov function are depicted in Figure 4.1. As can be seen, the Lyapunov function (with matrices P_1 and P_2) is defined only in the moments when there is a switching in the system: from $A_{1,z}$ to $A_{2,z}$ or from $A_{2,z}$ to $A_{1,z}$, respectively. A 1-sample variation of the Lyapunov function corresponds to the difference between two consecutive values of the Lyapunov function. A 2-sample variation corresponds to the difference after 2 samples of the Lyapunov function, etc.

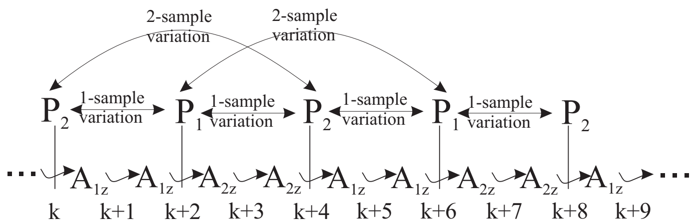


Figure 4.1: Switches in the system and in the Lyapunov function.

For the easier notation, we use a directed graph representation of the switching system (4.1), (4.2) or (4.3). The graph associated to a switching system is $\mathscr{G} = \{\mathscr{V}, \mathscr{E}\}$, with $\mathscr{V}$ being the set of vertices representing the subsystems and $\mathscr{E}$ the set of admissible transitions or switches. Thus, $(v_i, v_j) \in \mathscr{E}$ if a switch from subsystem i to subsystem j is possible. Note that we assume that self-transitions are also possible: these correspond to the subsystem being active for more than one sample. With a slight abuse of notation, we also use the notation $\mathscr{G} = \{\mathscr{V}, \mathscr{E}, \mathscr{W}\}$, with $\mathscr{W}$ being a matrix of weights associated to the vertices and edges. By convention, if a transition between two different subsystems is not possible, the corresponding weight is ∞.

A path $\mathscr{P}(v_i, v_j)$ between two vertices v_i and v_j in the graph $\mathscr{G}$ is a sequence of vertices $\mathscr{P}(v_i, v_j) = [v_{p_1}, v_{p_2}, \ldots, v_{p_{n_p}}]$ so that $v_i = v_{p_1}$, $v_j = v_{p_{n_p}}$, and $(v_{p_k}, v_{p(k+1)}) \in \mathscr{E}$, $k = 1, 2, \ldots, n_p - 1$. A path between two vertices is in general not unique. A cycle $\mathscr{C} = [c_1, c_2, \ldots, c_{n_c}, c_1]$ is a path having the same initial and final vertex. Two cycles are equivalent if the vertices in one are a cyclic permutation of the vertices in the other. When referring to paths and cycles, we mean elementary paths and cycles, i.e., paths in which each vertex may appear only once. A graph is strongly connected if there is a path between any two vertices in $\mathscr{V}$.

In a weighted graph $\mathscr{G} = \{\mathscr{V}, \mathscr{E}, \mathscr{W}\}$ we define the weight of a path $W(\mathscr{P}(v_i, v_j))$ as the product of all vertices and edges that appear in the path, i.e.,

$$W([v_{p_1}, v_{p_2}, \ldots, v_{p_{n_p}}]) = \prod_{k=1}^{n_p} w_{p_k, p_k} \cdot \prod_{k=1}^{n_p - 1} w_{p_k, p_{k+1}}$$

The weight of a cycle is similarly defined. A cycle is subunitary, if its weight is less than 1.

A path in a graph associated to a switching system corresponds to a switching law. A cycle in a graph associated to a switching system corresponds to a periodic switching law.

Let us illustrate the notions above on an example.

Example 4.1 Consider a switching system composed of four subsystems, $\boldsymbol{x}(k+1) = A_{i,z}\boldsymbol{x}(k)$, for $i = 1, 2, 3, 4$, and with admissible switches $(1,2)$, $(2,1)$, $(2,3)$, $(3,1)$, $(4,3)$, $(1,4)$. Next to this, each subsystem can be active for more than one sample. The corresponding graph representation is illustrated in Figure 4.2.

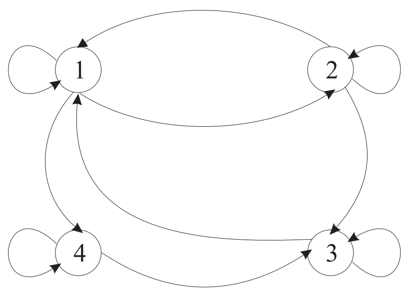


Figure 4.2: Graph representation of the switching system in Example 4.1.

The graph is $\mathscr{G} = \{\mathscr{V}, \mathscr{E}\}$, with $\mathscr{V} = \{1, 2, 3, 4\}$ and

$$\mathscr{E} = \{(1,1), (1,2), (1,4), (2,1), (2,2), (2,3), (3,1), (3,3), (4,3), (4,4)\}$$

Possible paths between vertices 1 and 3 are $\mathscr{P}(1,3) = [1, 2, 3]$ and $\mathscr{P}(1,3) = [1, 4, 3]$. The sequence $[1, 2, 1]$ is a cycle and is equivalent to $[2, 1, 2]$.

Let the associated weight matrix be given by

$$\mathscr{W} = \begin{pmatrix} 1 & 0.5 & \infty & 1 \\ 0.5 & 2 & 2 & \infty \\ 3 & \infty & 1 & \infty \\ \infty & \infty & 1 & 2 \end{pmatrix}$$

where ∞ corresponds to an inadmissible switch. The graph with the weights given in $\mathscr{W}$ is illustrated in Figure 4.3.

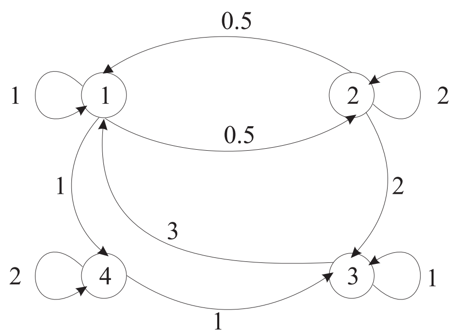


Figure 4.3: Graph representation of the switching system with weights in Example 4.1.

The weight of the path $\mathscr{P}(1,3) = [1, 2, 3]$ is $W(\mathscr{P}(1,3)) = w_{11}w_{12}w_{22}w_{23}w_{33} = 2$. The weight of the cycle $\mathscr{C} = [1, 2, 1]$ is $W(\mathscr{C}) = w_{11}w_{12}w_{22}w_{21} = 0.5 < 1$ so this cycle is subunitary. Since in the graph above there exists a path between any two vertices, the graph is strongly connected. $\square$

Once activated, a subsystem may be active continuously for at least $p_i^m \in \mathbb{N}^+$ and at most $p_i^M \in \mathbb{N}^+$ samples, that are assumed known.

4.4 Outline

The structure of this part is as follows. Chapter 5 presents results regarding stability analysis, Chapter 6 conditions for stabilization and Chapter 7 conditions for observer design. Stability analysis is performed for periodic systems and, using an α-sample variation of the Lyapunov function, for switching systems. Conditions for stabilization are presented for three cases: for periodic systems, for switching systems with any admissible switching law and for switching systems when the switching law can be chosen. Observer design is also considered for periodic systems, for switching systems and for the case when the switching law can be chosen. The conditions presented are discussed and illustrated on numerical examples in all cases.

Chapter 5

Stability analysis

This chapter considers stability analysis of the periodic or switching system (4.1), repeated here for convenience:

$$\begin{aligned} \boldsymbol{x}(k+1) &= \sum_{i=1}^{r_j} h_{ji}(\boldsymbol{z}_j(k)) A_{j,i} \boldsymbol{x}(k) \\ &= A_{j,z} \boldsymbol{x}(k) \end{aligned} \tag{5.1}$$

First, we present conditions which, when satisfied, ensure that the *periodic* system (5.1) is asymptotically stable. Second, we show conditions which ensure that the *switching* system (5.1) – i.e., the periodic assumption no longer holds – is asymptotically stable. In both cases, the conditions are extended to the α-sample variation of the Lyapunov function and discussed and illustrated on numerical examples.

5.1 Stability conditions for periodic systems

In this section, we consider the stability analysis of switching TS systems of the form (5.1). The results are also extended for α-sample variation.

5.1.1 Design conditions

Consider the periodic TS system (5.1), composed of n_s subsystems, with each subsystem j being active for p_j samples, $j = 1, 2, \ldots, n_\text{s}$. Then, the following results can be stated.

Theorem 5.1 *The periodic TS system* (5.1) *with periods* $p_1, p_2, \ldots, p_{n_\text{s}}$ *is asymptotically stable, if there exist* $P_{j,i} = P_{j,i}^T > 0$*,* $M_{j,i}$*,* $j = 1, 2, \ldots, n_\text{s}$*,* $i = 1, 2, \ldots, r_j$*, such* *that the following condition is satisfied:*

$$\begin{pmatrix} -P_{j,z} & (*) & \dots & (*) & (*) \\ M_{\underline{j+1},z}A_{\underline{j+1},z} & -M_{\underline{j+1},z}+(*) & \dots & (*) & (*) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & M_{\underline{j+1},z+p_{\underline{j+1}}-1}A_{\underline{j+1},z+p_{\underline{j+1}}-1} & \Omega_{j+1,j+1} \end{pmatrix} < 0 \tag{5.2}$$

for $j = 1, 2, \dots, n_{\mathrm{s}}$*, where* $\Omega_{j+1,j+1} = -M_{\underline{j+1},z+p_{\underline{j+1}}-1} + (*) + P_{\underline{j+1},z+p_{\underline{j+1}}}$.

Remark: Note that $\underline{j+1}$ is used because due to the periodicity the $n_{\mathrm{s}}+i$th subsystem is in fact the ith one.

Proof: By considering a periodic Lyapunov function *defined only in the instants when a switching takes place* in the system:

$$V = \boldsymbol{x}(k)^T P_{j,z}\boldsymbol{x}(k)$$

if the jth subsystem has been active, the difference in the Lyapunov function is

$$\begin{gathered} V(\boldsymbol{x}(k+p_{\underline{j+1}})) - V(\boldsymbol{x}(k)) = \\ \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+p_{\underline{j+1}}) \end{pmatrix}^T \begin{pmatrix} -P_{j,z} & 0 \\ 0 & P^{-1}_{\underline{j+1},z+p_{\underline{j+1}}} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+p_{\underline{j+1}}) \end{pmatrix} \end{gathered}$$

The closed-loop system dynamics during the $p_{\underline{j+1}}$ samples are

$$\begin{pmatrix} A_{j+1,1} & -I & \dots & 0 & 0 \\ 0 & A_{j+1,2} & -I & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & A_{j+1,p_{\underline{j+1}}-1} & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \\ \vdots \\ \boldsymbol{x}(k+p_{\underline{j+1}}) \end{pmatrix} = 0$$

Choosing $M = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ M_{\underline{j+1},z+1} & 0 & \dots & 0 & 0 \\ 0 & M_{\underline{j+1},z+2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & M_{\underline{j+1},z+p_{\underline{j+1}}-1} & 0 \\ 0 & 0 & \dots & 0 & M_{\underline{j+1},z+p_{\underline{j+1}}} \end{pmatrix}$ and applying Lemma 2.13 leads directly to (5.2). ■

Remark: Different from the control problem, LMI conditions may be obtained with virtually any multiplier matrix M, as long as the Lyapunov function uses $P_{j,z}$ instead of $P^{-1}_{j,z}$ as it is used in the control problem. However, to keep the notation simple, we only present the result for a specific case.

The result above can easily be extended using α-sample variation of the Lyapunov function defined in the switching instant. Note that this means that the Lyapunov function does not have to decrease in every instant where it is defined, but it should decrease every α instants. Recall that the Lypunov function is only defined in the switching instants, and the α-difference in the Lyapunov function corresponds to α consecutive switches in the system. Then, the following result can be formulated:

Theorem 5.2 *The periodic TS system* (5.1) *with periods* $p_1, p_2, \ldots, p_{n_s}$ *is asymptotically stable, if there exist* $P_{j,i} = P_{j,i}^T > 0$, $M_{j,l,i}$, $j = 1, 2, \ldots, n_s$, $i = 1, 2, \ldots, r_j$, $l = 1, 2, \ldots, \alpha$, *such that the following condition is satisfied:*

$$\begin{pmatrix} -P_{j,z} & (*) & \ldots & (*) & (*) & \ldots & (*) & (*) \\ \Gamma_{1,0} & -M_{\underline{j+1},1,z} + (*) & \ldots & (*) & (*) & \ldots & (*) & (*) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \Gamma_{2,p_{\underline{j+1}}} & -M_{\underline{j+1},2,z+p_{\underline{j+1}}} + (*) & \ldots & (*) & (*) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & \Gamma_{\alpha,t-1} & \Omega_{j+\alpha,j+\alpha} \end{pmatrix} < 0 \tag{5.3}$$

for $j = 1, 2, \ldots, n_s$, *where* $t = \sum_{i=1}^{\alpha} p_{\underline{j+i}}$, $\Omega_{j+\alpha,j+\alpha} = -M_{\underline{j+1},\alpha,z+t-1} - M_{\underline{j+1},\alpha,z+t-1}^T + P_{\underline{j+\alpha},z+t-1}$ *and* $\Gamma_{k,l} = M_{\underline{j+1},k,z+l} A_{\underline{j+k},z+l}$.

Proof: The proof is similar to that of Theorem 5.1. ■

Remark: More general conditions may be obtained by using a general matrix M in Lemma 2.13.

5.1.2 Examples and discussion

First, let us discuss how exactly the conditions derived in Section 5.1 are applied. For simplicity, consider a switching TS model consisting of two subsystems, each with period 2, similar to the one presented in Section 4.3.

For this system the conditions of Theorem 5.1 correspond to *there exist* $P_{j,i} = P_{j,i}^T > 0$, $M_{j,i}$, $j = 1, 2$, $i = 1, 2, \ldots, r_j$, *so that the following conditions are satisfied:*

$$\begin{aligned} &\begin{pmatrix} -P_{1,z} & (*) & (*) \\ M_{2,z}A_{2,z} & -M_{2,z} + (*) & (*) \\ 0 & M_{2,z+1}A_{2,z+1} & -M_{2,z+1} + (*) + P_{2,z+2} \end{pmatrix} < 0 \\ &\begin{pmatrix} -P_{2,z} & (*) & (*) \\ M_{1,z}A_{1,z} & -M_{1,z} + (*) & (*) \\ 0 & M_{1,z+1}A_{1,z+1} & -M_{1,z+1} + (*) + P_{1,z+2} \end{pmatrix} < 0 \end{aligned} \tag{5.4}$$

Relaxed LMI conditions can be formulated using Lemma 2.11, e.g.,

Corollary 5.3 *The system* (5.1) *is asymptotically stable if there exist* $P_{j,i} = P_{j,i}^T > 0$, $M_{j,i}$, $j = 1, 2$, $i = 1, 2, \dots, r_j$, *so that*

$$\frac{2}{r_{\underline{l+1}} - 1}\Gamma_{liimno} + \Gamma_{ljimno} + \Gamma_{lijmno} < 0$$

$i, j, l, m, n, o = 1, 2$, *where*

$$\Gamma_{lijmno} = \begin{pmatrix} -P_{l,i} & (*) & (*) \\ M_{\underline{l+1},i}A_{\underline{l+1},j} & -M_{\underline{l+1},i} + (*) & (*) \\ 0 & M_{\underline{l+1},m}A_{\underline{l+1},n} & -M_{\underline{l+1},m} + (*) + P_{\underline{l+1},o} \end{pmatrix}$$

Let us now consider a 2-sample variation of the Lyapunov function. The conditions of Theorem 5.2 become *there exist* $P_{j,i} = P_{j,i}^T > 0$, $M_{j,l,i}$, $j, l = 1, 2$, $i = 1, 2, \dots, r_j$, *so that the following conditions are satisfied:*

$$\begin{pmatrix} -P_{1,z} & (*) & (*) & (*) & (*) \\ M_{2,1,z}A_{2,z} & -M_{2,1,z} + (*) & (*) & (*) & (*) \\ 0 & M_{2,1,z+1}A_{2,z+1} & -M_{2,1,z+1} + (*) & (*) & (*) \\ 0 & 0 & M_{2,2,z+2}A_{1,z+2} & -M_{2,2,z+2} + (*) & (*) \\ 0 & 0 & 0 & M_{2,2,z+3}A_{1,z+3} & \begin{pmatrix} -M_{2,2,z+3}+ \\ (*) + P_{1,z+4} \end{pmatrix} \end{pmatrix} < 0$$

$$\begin{pmatrix} -P_{2,z} & (*) & (*) & (*) & (*) \\ M_{1,1,z}A_{1,z} & -M_{1,1,z} + (*) & (*) & (*) & (*) \\ 0 & M_{1,1,z+1}A_{1,z+1} & -M_{1,1,z+1} + (*) & (*) & (*) \\ 0 & 0 & M_{1,2,z+2}A_{2,z+2} & -M_{1,2,z+2} + (*) & (*) \\ 0 & 0 & 0 & M_{1,2,z+3}A_{2,z+3} & \begin{pmatrix} -M_{1,2,z+3}+ \\ (*) + P_{2,z+4} \end{pmatrix} \end{pmatrix} < 0$$

Similarly to the 1-sample variation, relaxed LMI conditions can be formulated.

Although the number of samples to be used in the variation of the Lyapunov function can be chosen as n_s, this is not necessary. On the other hand, by increasing α, the number and the dimension of the LMIs to be solved increases.

In developing the conditions, in Finsler's lemma we used the same matrices in the multiplication of the same subsystem, i.e., we have M_1A_1, M_2A_2, etc., even for different time instants. Different matrices can be used at each time instant, or even a completely general multiplier can be used in Lemma 2.13. This would significantly increase the number of decision variables. As it is, the number of decision variables is $\sum_{i=1}^{n_s} r_i n_x^2 + \sum_{i=1}^{n_s} r_i n_x(n_x + 1)/2$, where n_x denotes the dimension of the state. It should be noted that the number of decision variables depends on the number of the subsystems, the number of rules in each subsystem, and the dimension of the state. The number and the dimension of the actual LMIs to be solved depends on the relaxation used.

Note that the conditions do not require that the local matrices of the TS systems or even the individual subsystems are stable. We illustrate this on the following example.

Example 5.1 Consider the switching fuzzy system with two subsystems as follows:

$$\boldsymbol{x}(k+1) = \sum_{i=1}^{2} h_{1i}(\boldsymbol{z}_1(k)) A_{1i} \boldsymbol{x}(k)$$

with

$$A_{11} = \begin{pmatrix} -0.44 & -0.26 \\ -0.65 & 0.62 \end{pmatrix} \quad A_{12} = \begin{pmatrix} 1.1 & -0.2 \\ 0.53 & -0.27 \end{pmatrix}$$

with $h_{11} = exp(-x_1^2)$, $h_{12} = 1 - h_{11}$ and

$$\boldsymbol{x}(k+1) = \sum_{i=1}^{2} h_{2i}(\boldsymbol{z}_1(k)) A_{2i} \boldsymbol{x}(k)$$

with

$$A_{21} = \begin{pmatrix} 0.02 & 0.6 \\ -0.22 & -0.44 \end{pmatrix} \quad A_{22} = \begin{pmatrix} 0.32 & -0.15 \\ -1 & 0.8 \end{pmatrix}$$

with $h_{21} = cos(x_1)^2$, $h_{22} = 1 - h_{21}$.

The local models A_{12} and A_{22} are unstable, their eigenvalues being $\begin{pmatrix} 1.01 & -0.18 \end{pmatrix}$ and $\begin{pmatrix} 0.10 & 1.01 \end{pmatrix}$, respectively.

By switching between the two subsystems with a period $p_1 = 2$ for the first subsystem and $p_2 = 2$ for the second subsystem, the resulting periodic system is asymptotically stable, as illustrated in Figure 5.1.

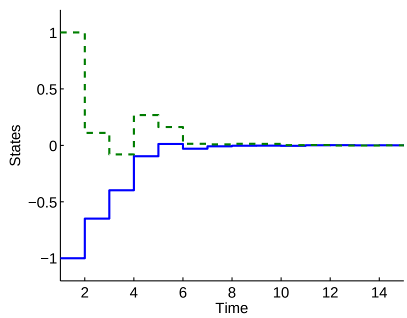


Figure 5.1: A trajectory of the states of the periodic system.

For this periodic system it is not possible to find either a quadratic or a non-quadratic Lyapunov function, common for both subsystems, as the corresponding LMIs are unfeasible.

Solving the conditions of Theorem 5.1 using the relaxation of (Wang et al., 1996), we obtain

$$P_{11} = \begin{pmatrix} 2.7195 & -0.6733 \\ -0.6733 & 3.7396 \end{pmatrix} \qquad P_{12} = \begin{pmatrix} 3.9735 & -1.7562 \\ -1.7562 & 3.6588 \end{pmatrix}$$
$$M_{11} = \begin{pmatrix} 4.4862 & 0.0610 \\ -0.0699 & 3.3215 \end{pmatrix} \qquad M_{12} = \begin{pmatrix} 4.2601 & 0.6030 \\ -0.3073 & 2.5510 \end{pmatrix}$$
$$P_{21} = \begin{pmatrix} 4.3030 & 0.0691 \\ 0.0691 & 2.1985 \end{pmatrix} \qquad P_{22} = \begin{pmatrix} 5.2906 & 0.0310 \\ 0.0310 & 1.9130 \end{pmatrix}$$
$$M_{21} = \begin{pmatrix} 3.9873 & -0.6752 \\ -0.6623 & 3.5612 \end{pmatrix} \qquad M_{22} = \begin{pmatrix} 3.5420 & -1.0998 \\ -0.7195 & 3.4535 \end{pmatrix}$$

Thus, the stability of the periodic system is proven. □

5.2 Analysis of switching systems

In the previous section, we have derived conditions by exploiting the fact that the subsystems are activated periodically. This section presents results for the case when the switching is not necessarily periodic.

5.2.1 Stability conditions

Lets us now consider the stability of the origin of the switching system

$$\boldsymbol{x}(k+1) = A_{j,z}\boldsymbol{x}(k) \tag{5.5}$$

Example 5.2 As an example, consider the system depicted in Figure 9.2. Switches are possible between subsystems 1 and 2 (in both directions), from 2 to 3, from 3 to 1, and from 3 to 3, i.e., subsystem 3 can remain continuously active. Consequently, $p_1^m = p_1^M = p_2^m = p_2^m = 1$, $p_3^m = 1$, $p_3^M = \infty$.

Due to the switching between the subsystems, a possibility to establish stability is to consider a switching Lyapunov function, and verify that it decreases during each subsystem and during each switch. This is in effect an extension to TS models of the results of (Daafouz et al., 2002), that was established for linear systems.

However, since subsystems 1 and 2 can only be active for one time instance, it can be seen that to guarantee the systems stability it is enough if the (switching) Lyapunov function decreases while subsystem 3 is active (since it can be continuously active), and it also decreases during the cycles $[1, 2, 1]$ and $[1,2,3,1]$. □

To derive stability conditions, consider the switching Lyapunov function

$$V = \boldsymbol{x}(k)^T P_{i,j,z}\boldsymbol{x}(k) \tag{5.6}$$

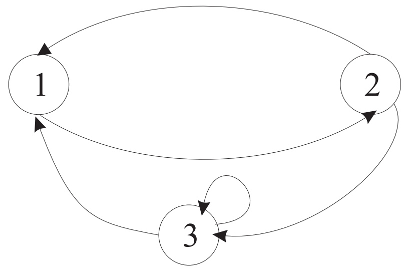


Figure 5.2: Switching system for Example 5.2.

defined during the switches, i.e., on the edges of the associated graph $\mathscr{G} = \{\mathscr{V}, \mathscr{E}\}$, with $(v_i, v_j) \in \mathscr{E}$.

Remark: If a subsystem i may be active for several number of samples, the edge (v_i, v_i) is also considered.

In a first step the following result can be formulated:

Theorem 5.4 *The origin of the switching system* (5.5) *is asymptotically stable, if there exist* $P_{i,j,k} = P_{i,j,k}^T > 0$, $M_{i,j,k}$, $(v_i, v_j) \in \mathscr{E}$, $k = 1, 2, \dots, r$, *such that*

$$\begin{pmatrix} -P_{i,j,z} & (*) \\ M_{j,l,z}A_{j,z} & -M_{j,l,z} - M_{j,l,z}^T + P_{j,l,z+1} \end{pmatrix} < 0 \tag{5.7}$$

for all admissible paths $\mathscr{P}(v_i, v_l) = [v_i, v_j, v_l]$, $v_i \in \mathscr{V}$.

Proof: Consider the switching Lyapunov function (5.6), defined on the edges of the associated graph, with $\boldsymbol{x}(k)^T P_{i,j,z}\boldsymbol{x}(k)$ being active during the transition from vertex i to vertex j. The difference in the Lyapunov function for two consecutive samples is

$$\begin{aligned} \Delta V &= \boldsymbol{x}(k+1)^T P_{j,l,z+1}\boldsymbol{x}(k+1) - \boldsymbol{x}(k)^T P_{i,j,z}\boldsymbol{x}(k) \\ &= \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -P_{i,j,z} & 0 \\ 0 & P_{j,l,z+1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} \end{aligned}$$

where $[v_i, v_j, v_l]$ is an admissible path.

During the transition for j to l, the dynamics of the system are described by

$$\begin{pmatrix} A_{j,z} & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} = 0$$

Using Lemma 2.13, the difference is the Lyapunov function is negative, if there exists M such that

$$\begin{pmatrix} -P_{i,j,z} & 0 \\ 0 & P_{j,l,z+1} \end{pmatrix} + M \begin{pmatrix} A_{j,z} & -I \end{pmatrix} + (*) < 0$$

By choosing

$$M = \begin{pmatrix} 0 \\ M_{j,l,z} \end{pmatrix}$$

we have directly (5.7). ■

Remark: It should be noted that in order to reduce the conservativeness of the solution and exploit available relaxations, a double sum can also be used in the matrix M, i.e., M can be chosen as

$$M = \begin{pmatrix} 0 \\ M_{j,l,z,z+1} \end{pmatrix}$$

leading to the following corollary:

Corollary 5.5 *The origin of the switching system* (5.5) *is asymptotically stable, if there exist* $P_{i,j,k} = P_{i,j,k}^T > 0$, $M_{i,j,k,\beta}$, $(v_i, v_j) \in \mathscr{E}$, $(v_j, v_l) \in \mathscr{E}$, $k, \beta = 1, 2, \ldots, r$, *such that*

$$\begin{pmatrix} -P_{i,j,z} & (*) \\ M_{j,l,z,z+1}A_{j,z} & -M_{j,l,z,z+1} - M_{j,l,z,z+1}^T + P_{j,l,z+1} \end{pmatrix} < 0 \tag{5.8}$$

for all admissible paths $\mathscr{P}(v_i, v_l) = [v_i, v_j, v_l]$, $v_i \in \mathscr{V}$.

In this case, using Lemma 2.11, we can formulate the conditions as LMIs, as follows:

Corollary 5.6 *The origin of the switching system* (5.5) *is asymptotically stable, if there exist* $P_{i,j,k} = P_{i,j,k}^T > 0$, $M_{i,j,k,l}$, $(v_i, v_j) \in \mathscr{E}$, $k, l = 1, 2, \ldots, r$, *such that*

$$\begin{aligned} &\Gamma_{kk}^{i,j,l} < 0 \\ &\frac{2}{r-1}\Gamma_{kk}^{i,j,l,\gamma} + \Gamma_{k\beta}^{i,j,l} + \Gamma_{\beta k}^{i,j,l} < 0, \quad k, \beta = 1, 2, \ldots, r \end{aligned}$$

with

$$\Gamma_{k\beta}^{i,j,l,\gamma} = \begin{pmatrix} -P_{i,j,k} & (*) \\ M_{j,l,k,\gamma}A_{j,\beta} & -M_{j,l,k,\gamma} + (*) + P_{j,l,\gamma} \end{pmatrix}$$

for all admissible paths $\mathscr{P}(v_i, v_l) = [v_i, v_j, v_l]$, $v_j \in \mathscr{V}$.

Let us now consider an α-sample variation of the Lyapunov function. As it has been proven by Kruszewski and Guerra (2007), it is not necessary that the Lyapunov function decreases every sample, but it is sufficient if it decreases every α samples, $\alpha > 1$. Considering the Lyapunov function (5.6), the following result can be formulated:

Theorem 5.7 *The origin of the switching system* (5.5) *is asymptotically stable, if there exist* $\alpha \in \mathbb{N}^+$, $P_{i,j,k} = P_{i,j,k}^T > 0$, $M_{i,j,k}$, $(v_i, v_j) \in \mathscr{E}$, $k = 1, 2, \ldots, r$, *such that*

$$\begin{pmatrix} -P_{i_1,i_2,z} & (*) & \ldots & (*) \\ M_{i_2,i_3,z}A_{i_2,z} & -M_{i_2,i_3,z} + (*) & \ldots & (*) \\ 0 & M_{i_3,i_4,z+1}A_{i_3,z+1} & \ldots & (*) \\ \vdots & \vdots & \ldots & \begin{pmatrix} M_{i_{\alpha+1},i_{\alpha+2},z+\alpha-1} + (*) \\ +P_{i_{\alpha+1},i_{\alpha+2},z+\alpha} \end{pmatrix} \end{pmatrix} < 0 \qquad (5.9)$$

for all admissible paths $\mathscr{P}(v_{i_1}, v_{i_{\alpha}+2}) = [v_{i_1}, v_{i_2}, \ldots, v_{i_{\alpha+2}}]$.

Proof: Consider the switching Lyapunov function (5.6), defined on the edges of the associated graph, with $P_{i,j,z}$ being active during the transition from vertex i to vertex j. The difference in the Lyapunov function for α consecutive samples is

$$\begin{aligned} \Delta V &= \boldsymbol{x}(k+\alpha)^T P_{i_{\alpha+1},i_{\alpha+2},z+\alpha} \boldsymbol{x}(k+\alpha) - \boldsymbol{x}(k)^T P_{i_1,i_2,z} \boldsymbol{x}(k) \\ &= \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+\alpha) \end{pmatrix}^T \begin{pmatrix} -P_{i_1,i_2,z} & 0 \\ 0 & P_{i_{\alpha+1},i_{\alpha+2},z+\alpha} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+\alpha) \end{pmatrix} \end{aligned}$$

where $[v_{i_1}, v_{i_2}, \ldots, v_{i_{\alpha+2}}]$ is an admissible path.

Along the switching sequence $[v_{i_1}, v_{i_2}, \ldots, v_{i_{\alpha+2}}]$, the dynamics of the system are described by

$$\begin{pmatrix} A_{i_2,z} & -I & 0 & \ldots & 0 \\ 0 & A_{i_3,z} & -I & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \\ \vdots \\ \boldsymbol{x}(k+\alpha) \end{pmatrix} = 0$$

Similarly to the proof of Theorem 5.4, using Lemma 2.13, the difference in the Lyapunov function is negative, if there exists M such that

$$\begin{pmatrix} -P_{i_1,i_2,z} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ldots \vdots & \\ 0 & 0 & \ldots & P_{i_{\alpha+1},i_{\alpha+2},z+\alpha} \end{pmatrix} + M \begin{pmatrix} A_{i_2,z} & -I & 0 & \ldots & 0 \\ 0 & A_{i_3,z} & -I & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -I \end{pmatrix} + (*) < 0$$

By choosing

$$M = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ M_{i_1,i_2,z} & 0 & \ldots & 0 \\ 0 & M_{i_2,i_3,z+1} & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & M_{i_{\alpha+1},i_{\alpha+2},z+\alpha} \end{pmatrix}$$

we have directly (5.9). ■

5.2.2 Discussion and examples

Let us now discuss the developed conditions, in particular those concerning the α-sample variation of the Lyapunov function, on an example.

Example 5.3 To illustrate the application of the conditions, let us revisit Example 5.2. The graph is $\mathscr{G} = \{\mathscr{V}, \mathscr{E}\}$, with $\mathscr{V} = \{1, 2, 3\}$ and

$$\mathscr{E} = \{(1,2), (2,1), (2,3), (3,1), (3,3)\}$$

The edge $(3,3)$ is introduced in order to take into account that subsystem 3 can be continuously active. The Lyapunov function is defined for every switch, i.e., we have $P_{i,2,z}$, $P_{2,1,z}$, $P_{2,3,z}$, etc. The conditions of Theorem 5.4 require that the Lyapunov function decreases with every switch/every sample, including here self-transitions, i.e., for all $(v_i, v_j) \in \mathscr{E}$. That is, we have the conditions:

$$\begin{aligned}
&\begin{pmatrix} -P_{1,2,z} & (*) \\ M_{2,1,z}A_{2,z} & -M_{2,1,z} + (*) + P_{2,1,z+1} \end{pmatrix} < 0 \\
&\begin{pmatrix} -P_{1,2,z} & (*) \\ M_{2,3,z}A_{2,z} & -M_{2,3,z} + (*) + P_{2,3,z+1} \end{pmatrix} < 0 \\
&\begin{pmatrix} -P_{2,3,z} & (*) \\ M_{3,3,z}A_{3,z} & -M_{3,3,z} + (*) + P_{3,3,z+1} \end{pmatrix} < 0 \\
&\begin{pmatrix} -P_{2,3,z} & (*) \\ M_{3,1,z}A_{3,z} & -M_{3,1,z} + (*) + P_{3,1,z+1} \end{pmatrix} < 0 \\
&\qquad\vdots
\end{aligned}$$

implying that each subsystem has to be stable. On the other hand, a 2-sample variation means that the Lyapunov function has to decrease along paths of lengths 3, i.e., we have the conditions:

$$\begin{aligned}
&\begin{pmatrix} -P_{1,2,z} & (*) & (*) \\ M_{2,1,z}A_{2,z} & -M_{2,1,z} + (*) & (*) \\ 0 & M_{1,2,z+1}A_{1,z+1} & -M_{1,2,z+1} + (*) + P_{1,2,z+2} \end{pmatrix} < 0 \\
&\begin{pmatrix} -P_{1,2,z} & (*) & (*) \\ M_{2,3,z}A_{2,z} & -M_{2,3,z} + (*) & (*) \\ 0 & M_{3,1,z+1}A_{3,z+1} & -M_{3,1,z+1} + (*) + P_{3,1,z+2} \end{pmatrix} < 0 \\
&\begin{pmatrix} -P_{1,2,z} & (*) & (*) \\ M_{2,3,z}A_{2,z} & -M_{2,3,z} + (*) & (*) \\ 0 & M_{3,3,z+1}A_{3,z+1} & -M_{3,3,z+1} + (*) + P_{3,3,z+2} \end{pmatrix} < 0 \\
&\vdots
\end{aligned}$$

Consider the following local models of the TS system above:

$$A_{1,1} = \begin{pmatrix} 0.23 & 0.2 \\ 0.03 & 0.9 \end{pmatrix} \qquad A_{1,2} = \begin{pmatrix} 0.47 & 0.47 \\ 0.23 & 0.16 \end{pmatrix}$$

$$A_{2,1} = A_{2,1} = \begin{pmatrix} 1.1 & 0 \\ 0.2 & 0.8 \end{pmatrix}$$

$$A_{3,1} = \begin{pmatrix} 0.11 & 0.09 \\ 0.09 & 0.07 \end{pmatrix} \qquad A_{3,2} = \begin{pmatrix} 0.32 & 0.30 \\ 0.09 & 0.38 \end{pmatrix}$$

The second subsystem is actually linear, but it is unstable. Due to this, available methods, including the extension of (Daafouz et al., 2002) to TS models yields unfeasible LMIs.

The LMIs given by Theorem 5.4 are also unfeasible. However, a 2-sample variation of the Lyapunov function (the conditions of Theorem 5.7 applied for $\alpha = 2$) results in a feasible set of LMIs. This is thanks to the fact that subsystem 2 can only be active for one sample, and therefore the Lyapunov function may decrease along a switching path of length at least 2. □

5.3 Conclusions

In this chapter we presented stability conditions for discrete-time periodic and switching systems represented by Takagi-Sugeno fuzzy models. The conditions have been developed using a periodic and a switching Lyapunov function defined on the switches. For periodic systems we have exploited the periodicity of the systems, while for switching systems we assumed that the switching sequence is not known in advance and it cannot be directly influenced. Thus, the conditions have to be solved for all admissible switches. The developed conditions have been formulated as LMIs and are able to prove stability of switching systems where one or more subsystems are unstable.

A shortcoming of the proposed conditions is the computational complexity of generating all the switching paths of length α, in particular for large α and large-scale switching systems and in consequence, the large number of LMIs that has to be solved. The LMI conditions can be relaxed by double sums, e.g., using $P_{i,j,z,z}$ instead of $P_{i,j,z}$, or even several sums in the Lyapunov function. Unfortunately, reducing the conservativeness of the conditions by introducing additional sums in the Lyapunov function (eventually leading to ANS conditions) also increases the number of LMIs.

Recall that we assume that the switching sequence is not known in advance and it cannot be directly influenced. With this assumption the stability conditions actually state that the switching system is stable if the Lyapunov function decreases along every path of length α. Naturally, this is the worst-case, i.e., all possible combinations on switches between the subsystems are taken into account. If the switching sequence can be chosen, or the goal is to find a stabilizing switching sequence, the conditions can be relaxed. Stabilization in such a way will be addressed in the next chapter.

Chapter 6

Stabilization of switching systems

This chapter considers stabilization of the periodic or switching system (4.2), repeated here for convenience:

$$\boldsymbol{x}(k+1) = A_{j,z}\boldsymbol{x}(k) + B_{j,z}\boldsymbol{u}(k) \tag{6.1}$$

First, we present conditions for designing a state-feedback control law, which, when satisfied, ensure that the *periodic* system (6.1) is asymptotically stable. Second, we show conditions for controller design such that the closed-loop *switching* system (6.1) – when the switches are not periodic and they cannot be influences – is asymptotically stable. Finally, under the assumption that the switching law can be chosen, we present conditions to chose a law that stabilizes the states of the system to zero. In all the cases the conditions are discussed and illustrated on numerical examples.

6.1 Controller design for periodic systems

6.1.1 Design conditions

In the previous chapter we have considered stability analysis of periodic systems. Now we extend the obtained results to controller design for periodic TS systems.

Consider the switching TS model (6.1), consisting of n_s subsystems, each having the period (being active for) p_i samples, $i = 1, 2 \ldots, n_\mathrm{s}$. For this system, we use the switching control input of the form

$$\boldsymbol{u}(k) = -F_{i,z}H_{i,z}^{-1}\boldsymbol{x}(k) \tag{6.2}$$

when the ith subsystem is active, $i = 1, 2, \ldots, n_\mathrm{s}$. The closed-loop dynamics are

$$\boldsymbol{x}(k+1) = (A_{i,z} - B_{i,z}F_{i,z}H_{i,z}^{-1})\boldsymbol{x}(k) \tag{6.3}$$

which is also a periodic system.

For (6.3) the following result can be stated:

Theorem 6.1 *The periodic TS system* (6.1) *with periods* $p_1, p_2, \ldots, p_{n_s}$ *is asymptotically stabilized by the control input* (6.2)*, if there exist* $P_{j,i} = P_{j,i}^T > 0$*,* $F_{j,i}$*,* $H_{j,i}$ $j = 1, 2, \ldots, n_s$*,* $i = 1, 2, \ldots, r_j$*, such that the following condition is satisfied:*

$$\begin{pmatrix} \begin{pmatrix} -H_{\underline{j+1},z} - H_{\underline{j+1},z}^T \\ +P_{j,z} \end{pmatrix} & (*) & \ldots & (*) & (*) \\ \Omega_{j+1,1} & -H_{\underline{j+1},z+1} + (*) & \ldots & (*) & (*) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \Omega_{j+1,p_{\underline{j+1}}} & -P_{\underline{j+1},z+p_{\underline{j+1}}} \end{pmatrix} < 0 \tag{6.4}$$

for $j = 1, 2, \ldots, n_s$*, where* $\Omega_{j+1,l} = A_{\underline{j+1},z+l-1} H_{\underline{j+1},z+l-1} - B_{\underline{j+1},z+l-1} F_{\underline{j+1},z+l-1}$.

Proof: Consider the switching Lyapunov function, defined only in the instants when a switching takes place in the system:

$$V = \boldsymbol{x}(k)^T P_{j,z}^{-1} \boldsymbol{x}(k)$$

if the active subsystem was j, $j = 1, 2 \ldots, n_s$.

The difference in the Lyapunov function is

$$\begin{aligned} & V(\boldsymbol{x}(k + p_{\underline{j+1}})) - V(\boldsymbol{x}(k)) = \\ & = \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k + p_{\underline{j+1}}) \end{pmatrix}^T \begin{pmatrix} -P_{j,z}^{-1} & 0 \\ 0 & P_{\underline{j+1},z+p_{\underline{j+1}}}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k + p_{\underline{j+1}}) \end{pmatrix} \end{aligned}$$

The closed-loop system dynamics during the $p_{\underline{j+1}}$ samples are

$$\begin{pmatrix} \Upsilon_{j+1,1} & -I & \ldots & 0 & 0 \\ 0 & \Upsilon_{j+1,2} & -I & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \Upsilon_{j+1,p_{\underline{j+1}}-1} & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \\ \vdots \\ \boldsymbol{x}(k + p_{\underline{j+1}}) \end{pmatrix} = 0$$

with $\Upsilon_{j+1,i} = A_{\underline{j+1},z+i} - B_{\underline{j+1},z+i} F_{\underline{j+1},z+i} H_{\underline{j+1},z+i}^{-1}$ for $i = 1, 2 \ldots, p_{\underline{j+1}} - 1$.

Choosing $M = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ H_{\underline{j+1},z+1}^{-T} & 0 & \ldots & 0 & 0 \\ 0 & H_{\underline{j+1},z+2}^{-T} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & H_{\underline{j+1},z+p_{\underline{j+1}}-1}^{T} & 0 \\ 0 & 0 & \ldots & 0 & P_{\underline{j+1},z+p_{\underline{j+1}}}^{-1} \end{pmatrix}$ and applying Lemma 2.13 leads to

$$\begin{pmatrix} -P_{jz}^{-1} & (*) & \dots & (*) & (*) \\ \Omega_{j+1,1} & -H_{\underline{j+1},z+1}^{-1}+(*) & \dots & (*) & (*) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \Omega_{j+1,p_{\underline{j+1}}} & -P_{j+1,z+p_{\underline{j+1}}}^{-1} \end{pmatrix} < 0$$

with $\Omega_{j+1,i} = H_{\underline{j+1},z+i}^{-T}(A_{\underline{j+1},z+i-1} - B_{\underline{j+1},z+i-1}F_{\underline{j+1},z+i-1}H_{\underline{j+1},z+i-1}^{-1})$ for $i = 1, 2, \dots, p_{\underline{j+1}} - 1$, $\Omega_{j+1,t} = P_{\underline{j+1},z+t}^{-T}(A_{\underline{j+1},z+t-1} - B_{\underline{j+1},z+t-1}F_{\underline{j+1},z+t-1}H_{\underline{j+1},z+t-1}^{-1})$.

Congruence with

$$\begin{pmatrix} H_{\underline{j+1},z}^{T} & 0 & \dots & 0 & 0 \\ 0 & H_{\underline{j+1},z+1}^{T} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & H_{\underline{j+1},z+p_{\underline{j+1}}-1}^{T} & 0 \\ 0 & 0 & \dots & 0 & P_{\underline{j+1},z+p_{\underline{j+1}}} \end{pmatrix}$$

and applying Property 5 leads directly to (6.4). ■

In what follows, we extend the above result to α-sample variation (Kruszewski et al., 2008). When α-sample variation of the Lyapunov function is considered, the following result can be stated.

Theorem 6.2 *The periodic TS system* (6.1) *with periods* $p_1, p_2, \dots, p_{n_s}$ *is asymptotically stabilized by the control input* (6.2)*, if there exist* $P_{j,i} = P_{j,i}^T > 0$*,* $F_{j,i}$*, and* $H_{j,i}$*,* $j = 1, 2, \dots, n_s$*,* $i = 1, 2, \dots, r_j$*,* $l = 1, 2, \dots, \alpha$*, such that the following condition is satisfied:*

$$\begin{pmatrix} \begin{pmatrix} -H_{\underline{j+1},z} - H_{\underline{j+1},z}^{T} \\ +P_{j,z} \end{pmatrix} & (*) & \dots & (*) & (*) \\ \Omega_{j+1,1} & -H_{\underline{j+1},z+1}+(*) & \dots & (*) & (*) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \Omega_{j+\alpha,p_{\underline{j+1}}} & -P_{\underline{j+\alpha},z+t} \end{pmatrix} < 0 \tag{6.5}$$

for $j = 1, 2, \dots, n_s$*, where* $\Omega_{j+i,l} = A_{\underline{j+i},z+l-1}H_{\underline{j+i},z+l-1} - B_{\underline{j+i},z+l-1}F_{\underline{j+i},z+l-1}$*,* $i = 1, 2, \dots, \alpha$*,* $l = 1, 2, \dots, t$*, and* $t = \sum_{i=1}^{\alpha} p_{\underline{j+i}}$*.*

Proof: The proof follows the same line as that of Theorem 6.1. ■

6.1.2 Examples and discussion

Let us discuss first how exactly the controller design conditions derived in Section 6.1.1 are applied. Consider a switching TS model consisting of two subsystems, each with period 2, and each having two rules, i.e., we have:

$$\boldsymbol{x}(k+1) = \begin{cases} \sum_{i=1}^{2} h_{1i}(\boldsymbol{z}_1(k))(A_{1i}\boldsymbol{x}(k) + B_{1i}\boldsymbol{u}(k)) & \text{if } k = 4m, 4m+1 \\ \sum_{i=1}^{2} h_{2i}(\boldsymbol{z}_2(k))(A_{2i}\boldsymbol{x}(k) + B_{2i}\boldsymbol{u}(k)) & \text{if } k = 4m+2, 4m+3 \end{cases} \tag{6.6}$$

For the system (6.6) the conditions of Theorem 6.1 correspond to *there exist* $P_{j,i} = P_{j,i}^T > 0$*,* $H_{j,i}$*, and* $F_{j,i}$*,* $j,i = 1,2$*, so that the following conditions are satisfied:*

$$\begin{aligned} &\begin{pmatrix} -H_{2,z} + (*) + P_{1,z} & (*) & (*) \\ A_{2,z}H_{2,z} - B_{2,z}F_{2,z} & -H_{2,z+1} + (*) & (*) \\ 0 & A_{2,z+1}H_{2,z+1} - B_{2,z+1}F_{2,z+1} & -P_{2,z+2} \end{pmatrix} < 0 \\ &\begin{pmatrix} -H_{1,z} + (*) + P_{2,z} & (*) & (*) \\ A_{1,z}H_{1,z} - B_{1,z}F_{1,z} & -H_{1,z+1} + (*) & (*) \\ 0 & A_{1,z+1}H_{1,z+1} - B_{1,z+1}F_{1,z+1} & -P_{1,z+2} \end{pmatrix} < 0 \end{aligned} \tag{6.7}$$

The conditions of Theorem 6.2, e.g., for a 2-sample variation become *there exist* $P_{j,i} = P_{j,i}^T > 0$*,* $H_{j,i}$*, and* $F_{j,i}$*,* $j,i = 1,2$*, so that the following conditions are satisfied:*

$$\begin{pmatrix} -H_{2,z} + (*) + P_{1,z} & (*) & (*) & (*) & (*) \\ \Omega_{21} & -H_{2,z+1} + (*) & (*) & (*) & (*) \\ 0 & \Omega_{22} & -H_{1,z+2} + (*) & (*) & (*) \\ 0 & 0 & \Omega_{13} & -H_{1,z+3} + (*) & (*) \\ 0 & 0 & 0 & \Omega_{14} & -P_{1,z+4} \end{pmatrix} < 0$$

$$\begin{pmatrix} -H_{1,z} + (*) + P_{2,z} & (*) & (*) & (*) & (*) \\ \Omega_{11} & -H_{1,z+1} + (*) & (*) & (*) & (*) \\ 0 & \Omega_{12} & -H_{2,z+2} + (*) & (*) & (*) \\ 0 & 0 & \Omega_{23} & -H_{2,z+3} + (*) & (*) \\ 0 & 0 & 0 & \Omega_{24} & -P_{2,z+4} \end{pmatrix} < 0$$

$$\Omega_{ji} = A_{j,z+i-1}H_{j,z+i-1} - B_{j,z+i-1}F_{j,z+i-1}$$

As illustrated in the above conditions, for each subsystem we have a matrix inequality. In the ith matrix inequality, the first line corresponds to $P_{i,z}$, the next p_{i+1} lines correspond to the $i+1$th subsystems, etc.

As can be expected based on the stability conditions, in order for the closed-loop switching system to be stable it is not necessary that each subsystem is stabilized. We illustrate this on the following example, adopted from Kruszewski and Guerra (2007).

Example 6.1 Consider the periodic fuzzy system, composed of two subsystems, with the local matrices:

$$A_{11} = \begin{pmatrix} 1.5 & 10 \\ 0 & 0.5 \end{pmatrix} \qquad A_{12} = \begin{pmatrix} 0.5 & 10 \\ 0 & 0 \end{pmatrix}$$

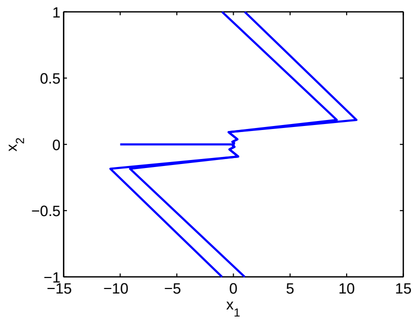


Figure 6.1: Trajectories of the closed-loop system – Example 6.1.

$$A_{21} = \begin{pmatrix} 1+a & 1 \\ 0 & 0.5 \end{pmatrix} \qquad A_{22} = \begin{pmatrix} 1 & 10 \\ 0 & 0.5 \end{pmatrix}$$

$$B_{11} = B_{12} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad B_{21} = B_{22} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

where ‘a’ is a real-valued parameter.

The control design for this system cannot be performed using a Lyapunov function that is common for both subsystems, be that quadratic or nonquadratic. In the switching system above, the first subsystem is not stabilizable, as $B_{11} = B_{12} = 0$, and the first local model A_{11} is unstable.

With the approach of Kruszewski and Guerra (2007), the maximum interval for ‘a’ that can be obtained is $[-250, 250]$. Using the conditions of Theorem 6.1, we obtain for $a = 1500$

$$H_{11} = \begin{pmatrix} 240.8746 & -33.2523 \\ -34.5795 & 10.3966 \end{pmatrix} \qquad H_{12} = \begin{pmatrix} 239.2061 & -37.6113 \\ -11.2734 & 7.4435 \end{pmatrix}$$

$$H_{21} = \begin{pmatrix} 374.8946 & 8.7529 \\ 0.0039 & 33.2571 \end{pmatrix} \qquad H_{22} = \begin{pmatrix} 374.8945 & 8.7468 \\ -0.0039 & 33.2570 \end{pmatrix}$$

$$P_{11} = \begin{pmatrix} 561.3074 & 8.7531 \\ 8.7531 & 18.2803 \end{pmatrix} \qquad P_{12} = \begin{pmatrix} 561.3065 & 8.7466 \\ 8.7466 & 18.2802 \end{pmatrix}$$

$$P_{21} = \begin{pmatrix} 295.0400 & -45.1654 \\ -45.1654 & 9.8090 \end{pmatrix} \qquad P_{22} = \begin{pmatrix} 300.2630 & -38.8747 \\ -38.8747 & 7.7637 \end{pmatrix}$$

$$F_{21} = 10^5 \begin{pmatrix} 5.6272 & 0.1325 \end{pmatrix} \qquad F_{22} = \begin{pmatrix} 374.8512 & 417.8578 \end{pmatrix}$$

which stabilizes the system. Moreover, the value of ‘a’ can still be increased. Trajectories of the closed-loop system for $a = 1500$ are shown in Figure 6.1. The membership functions used were $h_{11} = exp(-x_1^2)$, $h_{12} = 1 - h_{11}$ and $h_{21} = cos(x_1)^2$, $h_{22} = 1 - h_{21}$. □

To illustrate the efficiency of the proposed method, consider now a more complex example.

Example 6.2 Consider the periodic fuzzy system, composed of 5 subsystems, with the local matrices:

$$
\begin{aligned}
A_{11} &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & A_{12} &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \\
B_{11} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} & B_{12} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
A_{21} &= \begin{pmatrix} 1.5 & 1 \\ 0 & 1.5 \end{pmatrix} & A_{22} &= \begin{pmatrix} 1.5 & 1 \\ 0 & 1.5 \end{pmatrix} \\
B_{21} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} & B_{22} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
A_{31} &= \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} & A_{32} &= \begin{pmatrix} 1.5 & 0 \\ 0 & .5 \end{pmatrix} \\
B_{31} &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} & B_{32} &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
A_{41} &= \begin{pmatrix} -0.44 & -0.26 \\ -0.65 & 0.62 \end{pmatrix} & A_{42} &= \begin{pmatrix} 1.1 & -0.2 \\ 0.53 & -0.27 \end{pmatrix} \\
B_{41} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} & B_{42} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
A_{51} &= \begin{pmatrix} 1.32 & -0.15 \\ -1 & 0.8 \end{pmatrix} & A_{52} &= \begin{pmatrix} 0.02 & 1.6 \\ -0.22 & -0.44 \end{pmatrix} \\
B_{51} &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} & B_{52} &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\end{aligned}
$$

and periodicity $p = [1, 1, 1, 2, 2]$.

Again, several local models are not controllable and not stable, and consequently the control design cannot be performed using a Lyapunov function that is common for both subsystems, be that quadratic or nonquadratic. However, by using a switching control law, the system is stabilized. A trajectory of the closed-loop system is shown in Figure 6.2. To design the controller, 18 4×4 and 36 6×6 LMIs have been solved, and 90 decision variables have been computed. □

Relaxed LMI conditions can easily be obtained using e.g., Lemma 2.11. The number of decision variables in the LMIs for the 1-sample variation is $\sum_{i=1}^{n_s} r_i(n_x^2 + n_x(n_x+1)/2 + n_x n_u)$, where n_x denotes the dimension of the state and n_u the dimension of the input.

Compared to the possibilities in case of stability conditions, the freedom in choosing the multiplier in Lemma 2.13 is much smaller. This is because the controller is

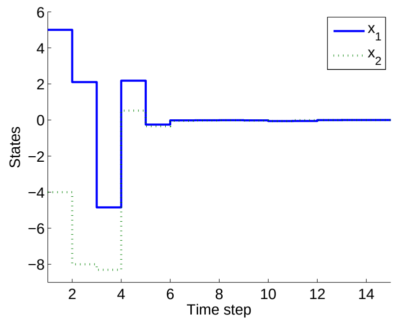


Figure 6.2: A trajectory of the closed-loop system – Example 6.2.

fixed for each subsystem. Also in this case, the number of decision variables depends only on the number of the subsystems, the number of rules in each subsystem, and the dimension of the state and input. The number and the dimension of the actual LMIs to be solved depends on the relaxation used and on the sample-variation used, as by increasing α, the number and dimension of the LMIs to be solved increases.

Let us illustrate this on the following example:

Example 6.3 Consider a periodic fuzzy system consisting of 3 subsystems, each with 2 rules, 2 states, and 1 input. A comparison of the number and dimension of the LMIs to be solved for controller design for different periodicity of the subsystems and different sample variation, using the relaxation of Wang et al. (1996) on all possible pairs is given in Table 6.1. The number of decision variables is $\sum_{i=1}^{n_s} r_i(n_x^2 + n_x(n_x + 1)/2 + n_x n_u) = 54$, in all the cases.

As can be seen, with a higher α-sample variation used, the number of LMIs and their dimension quickly increases. □

6.2 Controller design for switching systems

In this section, we present relaxed LMI conditions for the stabilization of switching TS systems. Two different switching nonquadratic Lyapunov functions are used and therefore two different sets of conditions are developed. We assume that the set of the admissible switches is known, but the exact switching sequence is not known in advance. This is a worst-case assumption. However, by taking into account the admissible switches, it is possible to develop conditions for the stabilization of some systems with uncontrollable local models.

Table 6.1: Comparison of number of LMIs

Periods	Samples	Number	Dimension
1, 1, 1	1	18	4×4
1, 1, 2	1	12	4×4
		18	6×6
1, 2, 2	1	36	6×6
		6	4×4
2, 2, 2	1	54	6×6
2, 2, 3	1	36	6×6
		54	8×8
2, 3, 3	1	18	6×6
		108	8×8
1, 1, 1	2	54	6×6
1, 1, 2	2	18	6×6
		108	8×8
1, 2, 2	2	162	10×10
		108	8×8
2, 2, 2	2	486	10×10
1, 1, 1	3	162	8×8
1, 1, 2	3	486	10×10

6.2.1 Design conditions

Recall that the switching control law we consider is

$$\boldsymbol{u}(k) = -F_{i,z}H_{i,z}^{-1}\boldsymbol{x}(k) \tag{6.8}$$

and the closed-loop system is expressed as

$$\boldsymbol{x}(k+1) = (A_{i,z} - B_{i,z}F_{i,z}H_{i,z}^{-1})\boldsymbol{x}(k) \tag{6.9}$$

In what follows, we use two switching Lyapunov functions:

$$V = \boldsymbol{x}(k)^T P_{i,j,z}^{-1}\boldsymbol{x}(k) \tag{6.10}$$

and

$$V = \boldsymbol{x}(k)^T H_{i,z}^{-T} P_{i,j,z} H_{i,z}^{-1}\boldsymbol{x}(k) \tag{6.11}$$

respectively, defined during the switches, i.e., on the edges of the associated graph $\mathscr{G} = \{\mathscr{V}, \mathscr{E}\}$, with $(v_i, v_j) \in \mathscr{E}$. The subscript indices i, j denote that the corresponding Lyapunov function is active if we switch from subsystem i to subsystem j.

Let us consider first (6.10). The difference in the Lyapunov function is

$$\begin{aligned}\Delta V &= \boldsymbol{x}(k+1)^T P_{j,l,z+1}^{-1} \boldsymbol{x}(k+1) - \boldsymbol{x}(k)^T P_{i,j,z}^{-1} \boldsymbol{x}(k) \\ &= \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -P_{i,j,z}^{-1} & 0 \\ 0 & P_{j,l,z+1}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}\end{aligned}$$

where $[v_i, v_j, v_l]$ is an admissible path.

Remark: If a subsystem i may be active for several samples, the Lyapunov function above is in fact used to prove its stability. However, if a subsystem is active for only one sample, it is not necessary for it to be stable.

On the edge $[v_i, v_j]$, the dynamics of the system are described by

$$\begin{pmatrix} A_{i,z} - B_{i,z} F_{i,z} H_{i,z}^{-1} & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} = 0$$

Using Lemma 2.13, the difference is the Lyapunov function is negative, if there exists M such that

$$M \begin{pmatrix} A_{i,z} - B_{i,z} F_{i,z} H_{i,z}^{-1} & -I \end{pmatrix} + (*) + \begin{pmatrix} -P_{i,j,z}^{-1} & 0 \\ 0 & P_{j,l,z+1}^{-1} \end{pmatrix} < 0$$

By choosing

$$M = \begin{pmatrix} 0 \\ P_{j,l,z+1}^{-1} \end{pmatrix}$$

we have

$$\begin{pmatrix} -P_{i,j,z}^{-1} & (*) \\ P_{j,l,z+1}^{-1} A_{i,z} - P_{j,l,z+1}^{-1} B_{i,z} F_{i,z} H_{i,z}^{-1} & -P_{j,l,z+1}^{-1} \end{pmatrix} < 0$$

Congruence with

$$\begin{pmatrix} H_{i,z}^T & 0 \\ 0 & P_{j,l,z+1} \end{pmatrix}$$

leads to

$$\begin{pmatrix} -H_{i,z}^T P_{i,j,z}^{-1} H_{i,z} & (*) \\ A_{i,z} H_{i,z} - B_{i,z} F_{i,z} & -P_{j,l,z+1} \end{pmatrix} < 0$$

and applying Proposition 5 we have

$$\begin{pmatrix} -H_{i,z} - H_{i,z}^T + P_{i,j,z} & (*) \\ A_{i,z} H_{i,z} - B_{i,z} F_{i,z} & -P_{j,l,z+1} \end{pmatrix} < 0$$

The condition developed above can be formulated as follows.

Theorem 6.3 *The closed-loop system* (6.9) *is asymptotically stable if there exist matrices $H_{i,j}$ and symmetric positive definite matrices $P_{i,j,m} = P_{i,j,m}^T > 0$, $i, j = 1, 2, \ldots, n_s$, $(v_i, v_j) \in \mathscr{E}$, $(v_j, v_l) \in \mathscr{E}$, $m, n = 1, 2, \ldots, r_i$, $o = 1, 2, \ldots, r_j$ so that Lemma 2.11 holds with*

$$\Gamma_{m,n,o}^{i,j,l} = \begin{pmatrix} -H_{i,m} - H_{i,m}^T + P_{i,j,m} & (*) \\ A_{i,n}H_{i,m} - B_{i,n}F_{i,m} & -P_{j,l,o} \end{pmatrix}$$

Let us now consider (6.11). The difference in the Lyapunov function is

$$\begin{aligned} \Delta V &= \boldsymbol{x}(k+1)^T H_{j,z+1}^{-T} P_{j,l,z+1} H_{j,z+1}^{-1} \boldsymbol{x}(k+1) - \boldsymbol{x}(k)^T H_{i,z}^{-T} P_{i,j,z} H_{i,z}^{-1} \boldsymbol{x}(k) \\ &= \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -H_{i,z}^{-T} P_{i,j,z} H_{i,z}^{-1} & 0 \\ 0 & H_{j,z+1}^{-T} P_{j,l,z+1} H_{j,z+1}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} \end{aligned}$$

where $[v_i, v_j, v_l]$ is an admissible path.

On the edge $[v_i, v_j]$, similarly to the previous case, the dynamics of the system are described by

$$\begin{pmatrix} A_{i,z} - B_{i,z}F_{i,z}H_{i,z}^{-1} & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} = 0$$

Using Lemma 2.13, the difference is the Lyapunov function is negative, if there exists M such that

$$M \begin{pmatrix} A_{i,z} - B_{i,z}F_{i,z}H_{i,z}^{-1} & -I \end{pmatrix} + (*) + \begin{pmatrix} -H_{i,z}^{-T} P_{i,j,z} H_{i,z}^{-1} & 0 \\ 0 & H_{j,z+1}^{-T} P_{j,l,z+1} H_{j,z+1}^{-1} \end{pmatrix} < 0$$

By choosing

$$M = \begin{pmatrix} 0 \\ H_{j,z+1}^{-1} \end{pmatrix}$$

we have

$$\begin{pmatrix} -H_{i,z}^{-T} P_{i,j,z} H_{i,z}^{-1} & (*) \\ H_{j,z+1}^{-1} A_{i,z} - H_{j,z+1}^{-1} B_{i,z} F_{i,z} H_{i,z}^{-1} & \begin{pmatrix} -H_{j,z+1}^{-T} + (*) + \\ H_{j,z+1}^{-T} P_{j,l,z+1} H_{j,z+1}^{-1} \end{pmatrix} \end{pmatrix} < 0$$

Congruence with

$$\begin{pmatrix} H_{i,z}^T & 0 \\ 0 & H_{j,z+1}^T \end{pmatrix}$$

leads to

$$\begin{pmatrix} -P_{i,j,z} & (*) \\ A_{i,z}H_{i,z} - B_{i,z}F_{i,z} & -H_{j,z+1} - H_{j,z+1}^T + P_{j,l,z+1} \end{pmatrix} < 0$$

The condition developed above can be formulated as follows.

Theorem 6.4 *The closed-loop system* (6.9) *is asymptotically stable if there exist matrices $H_{i,j}$ and symmetric positive definite matrices $P_{i,j,m} = P_{i,j,m}^T > 0$, $i, j = 1, 2, \ldots, n_s$, $(v_i, v_j) \in \mathscr{E}$, $(v_j, v_l) \in \mathscr{E}$, $m, n = 1, 2, \ldots, r_i$, $o = 1, 2, \ldots, r_j$, so that Lemma 2.11 holds with*

$$\Gamma_{m,n,o}^{i,j,l} = \begin{pmatrix} -P_{i,j,m} & (*) \\ A_{i,m}H_{i,n} - B_{i,m}F_{i,n} & -H_{j,o} - H_{j,o}^T + P_{j,l,o} \end{pmatrix}$$

6.2.2 Example and discussion

Let us discuss the previous conditions on an example.

Example 6.4 To illustrate the application of the conditions, consider a switching TS system composed of three subsystems, each having two local models. The switching graph is defined as $\mathscr{G} = \{\mathscr{V}, \mathscr{E}\}$, with $\mathscr{V} = \{1, 2, 3\}$ and $\mathscr{E} = \{(1,1), (1,2), (2,3), (3,1), (3,2)\}$. The edge $(1,1)$ is introduced in order to take into account that subsystem 1 can be active for several samples. The graph is illustrated in Figure 6.3.

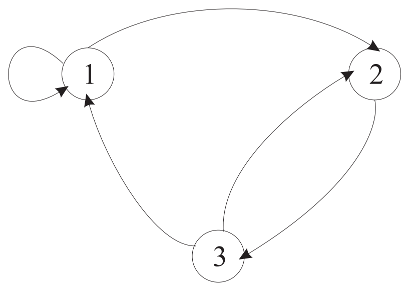


Figure 6.3: Graph representation of the switching system in Example 6.4.

The Lyapunov functions are defined for all the possible switches, i.e., we have $P_{1,1,z}$, $P_{1,2,z}$, $P_{2,3,z}$, etc. Consider the following local models of the switching TS system above:

$$A_{1,1} = \begin{pmatrix} 0.60 & -1.02 \\ 0.94 & -0.07 \end{pmatrix} \qquad B_{1,1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A_{1,2} = \begin{pmatrix} 0.08 & -1.78 \\ -1.77 & -0.66 \end{pmatrix} \qquad B_{1,2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A_{2,1} = \begin{pmatrix} 1.35 & 0.16 \\ 2.13 & -1.70 \end{pmatrix} \qquad B_{2,1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$A_{2,2} = \begin{pmatrix} 0.27 & -0.09 \\ 0.39 & 0.17 \end{pmatrix} \qquad B_{2,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$A_{3,1} = \begin{pmatrix} -1.83 & 0.81 \\ -1.50 & -0.23 \end{pmatrix} \qquad B_{3,1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$A_{3,2} = \begin{pmatrix} -1.63 & -0.79 \\ -0.31 & 0.69 \end{pmatrix} \qquad B_{3,2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For this particular example, both Theorems 6.3 and 6.4 provide stabilizing control laws. For instance, using Theorem 6.3 we obtain the gain matrices

$$H_{1,1} = \begin{pmatrix} 0.21 & -0.01 \\ 0.00 & 0.32 \end{pmatrix} \qquad F_{1,1} = \begin{pmatrix} 0.09 & -0.40 \end{pmatrix}$$
$$H_{1,2} = \begin{pmatrix} 0.11 & -0.08 \\ -0.03 & 0.34 \end{pmatrix} \qquad F_{1,2} = \begin{pmatrix} 0.05 & -0.56 \end{pmatrix}$$
$$H_{2,1} = \begin{pmatrix} 0.05 & 0.04 \\ 0.04 & 0.33 \end{pmatrix} \qquad F_{2,1} = \begin{pmatrix} 0.17 & -0.15 \end{pmatrix}$$
$$H_{2,2} = \begin{pmatrix} 0.07 & 0.07 \\ 0.04 & 0.34 \end{pmatrix} \qquad F_{2,2} = \begin{pmatrix} 0.14 & 0.04 \end{pmatrix}$$
$$H_{3,1} = \begin{pmatrix} 0.14 & -0.11 \\ -0.16 & 0.44 \end{pmatrix} \qquad F_{3,1} = \begin{pmatrix} -0.40 & 0.51 \end{pmatrix}$$
$$H_{3,2} = \begin{pmatrix} 0.11 & -0.21 \\ -0.22 & 0.51 \end{pmatrix} \qquad F_{3,2} = \begin{pmatrix} -0.30 & 0.67 \end{pmatrix}$$

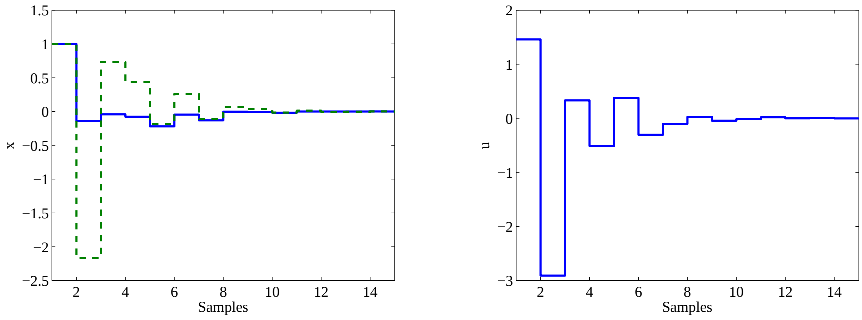
A trajectory of the closed-loop system, with initial states $\boldsymbol{x}(0) = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$ is presented in Figure 6.4(a) while the corresponding control input is presented in Figure 6.4(b). The switching sequence was 1 1 2 3 2 3 1 2 3 1 1 2 3 2 3. The membership functions used are the following ones:

$$h_{1,1}(\boldsymbol{x}) = \frac{1}{2}(1 - \sin(x_1)) \qquad h_{1,2}(\boldsymbol{x}) = 1 - h_{1,1}(\boldsymbol{x})$$
$$h_{2,1}(\boldsymbol{x}) = \frac{1}{2}(1 - \cos(x_1)) \qquad h_{2,2}(\boldsymbol{x}) = 1 - h_{2,1}(\boldsymbol{x})$$
$$h_{3,1}(\boldsymbol{x}) = \frac{1}{2}(1 - e^{-x_1^2}) \qquad h_{3,2}(\boldsymbol{x}) = 1 - h_{3,1}(\boldsymbol{x})$$

□

Although for the example above the conditions of both Theorems 6.3 and 6.4 are feasible, it has to be noted that in general, the two sets of conditions are not equivalent.

By considering the switching possibilities in the Lyapunov matrix $P_{i,j,z}$, the developed conditions extend the case of using switching nonquadratic Lyapunov functions, i.e., simply Lyapunov matrices of the form $P_{i,z}$, for the switching system. This can be easily seen by inspecting the LMI conditions. Indeed, if there exists a solution using a switching Lyapunov function, then the conditions of Theorem 6.3 or 6.4 are satisfied.



(a) Closed-loop trajectory. (b) Control input

Figure 6.4: Simulation results for Example 6.4.

However, the reverse is not true. For instance, for Example 6.4, the LMI conditions corresponding to the use of a simple switching Lyapunov function in Theorem 6.4, i.e., the conditions

$$\frac{2}{r-1}\Gamma^{i,i}_{m,n,o} + \Gamma^{i,j}_{m,n,o} + \Gamma^{j,i}_{m,n,o} < 0$$

$i, j = 1, 2, \dots, n_s$, $(v_i, v_j) \in \mathscr{E}$, $m, n = 1, 2, \dots, r_i$, $o = 1, 2, \dots, r_j$ with

$$\Gamma^{i,j}_{m,n,o} = \begin{pmatrix} -P_{i,m} & (*) \\ A_{i,m}H_{i,n} - B_{i,m}F_{i,n} & -H_{j,o} - H^T_{j,o} + P_{j,o} \end{pmatrix}$$

are not feasible.

Recall that we assume that the switching sequence is not known in advance and it is not assumed to be directly influenced. Naturally, this is the worst-case, i.e., all possible combinations on switches between the subsystems are taken into account. If the switching sequence can be chosen, or the goal is to find a stabilizing switching sequence, the conditions can be relaxed. The LMI conditions can also be relaxed by double sums in the Lyapunov function, e.g., using $P_{i,j,z,z}$ instead of $P_{i,j,z}$, or even several sums. Such relaxations are presented in the following section.

6.2.3 Extensions

By looking at the Lyapunov matrices, a straightforward extension of the results above is by using the control law

$$\boldsymbol{u}(k) = -F_{i,j,z}H^{-1}_{i,j,z}\boldsymbol{x}(k) \tag{6.12}$$

i.e., instead of choosing a control law to be applied for each subsystem, the control is applied based on the switching that takes place. Using the Lyapunov function (6.10) and (6.11), respectively, the following results can be formulated. For the Lyapunov function (6.10) we have

Corollary 6.5 *The switching TS system* (6.1) *is asymptotically stabilized by the switching control law* (6.12) *if there exist matrices* $H_{i,j,m}$ *and symmetric positive definite matrices* $P_{i,j,m} = P_{i,j,m}^T > 0$, $i, j = 1, 2, \ldots, n_s$, $(v_i, v_j) \in \mathscr{E}$, $(v_j, v_l) \in \mathscr{E}$, $m, n = 1, 2, \ldots, r_i$, *so that*

$$\begin{pmatrix} -H_{i,j,z} - H_{i,j,z}^T + P_{i,j,z} & (*) \\ A_{i,z}H_{i,j,z} - B_{i,z}F_{i,j,z} & -P_{j,l,z+1} \end{pmatrix} < 0$$

while when using the Lyapunov function (6.11) we obtain

Corollary 6.6 *The switching TS system* (6.1) *is asymptotically stabilized by the switching control law* (6.12) *if there exist matrices* $H_{i,j,m}$ *and symmetric positive definite matrices* $P_{i,j,m} = P_{i,j,m}^T > 0$, $i, j = 1, 2, \ldots, n_s$, $(v_i, v_j) \in \mathscr{E}$, $(v_j, v_l) \in \mathscr{E}$, $m, n = 1, 2, \ldots, r_i$, *so that*

$$\begin{pmatrix} -P_{i,j,z} & (*) \\ A_{i,z}H_{i,j,z} - B_{i,z}F_{i,j,z} & -H_{j,l,z+1} + (*) + P_{j,l,z+1} \end{pmatrix} < 0$$

The proofs of the above corollaries follow the same lines as Theorems 6.3 and 6.4 and are therefore not repeated here. Similarly to the previous results, LMI conditions can be formulated using Lemmas 2.10 or 2.11. Unfortunately, the main drawback of this extension is that the switching sequence must be known in advance or directly dependent on the input.

A different possibility to extend the results is the use of delayed controller and delayed Lyapunov function (Lendek et al., 2012), as follows. Consider instead of the control law (6.8) the following

$$\boldsymbol{u}(k) = -F_{i,z-1,z}H_{i,z-1,z}^{-1}\boldsymbol{x}(k) \tag{6.13}$$

which depends not only on the current, but also on the past states through the evaluation of the scheduling variable at time $k-1$. To develop and relax the conditions, also instead of the Lyapunov functions (6.10) and (6.11), the delayed Lyapunov functions

$$V = \boldsymbol{x}(k)^T P_{i,j,z-1}^{-1}\boldsymbol{x}(k) \tag{6.14}$$

and

$$V = \boldsymbol{x}(k)^T H_{i,z,z-1}^{-T} P_{i,j,z,z-1} H_{i,z,z-1}^{-1}\boldsymbol{x}(k) \tag{6.15}$$

respectively, can be used, again defined during the switches. Then, based on the same steps as described in the previous section, the following results can be stated. Using the Lyapunov function (6.14), we have

Corollary 6.7 *The switching TS system* (6.1) *is asymptotically stabilized by the switching control law* (6.13) *if there exist matrices* $H_{i,j,m,n}$ *and symmetric positive definite* *matrices* $P_{i,j,m,n} = P_{i,j,m,n}^T > 0,\ i,j = 1,2,\ldots,n_s,\ (v_i,v_j) \in \mathscr{E},\ (v_j,v_l) \in \mathscr{E},\ m,n = 1,2,\ldots,r_i$, *so that*

$$\begin{pmatrix} -H_{i,j,z,z-1} - H_{i,j,z,z-1}^T + P_{i,j,z-1} & (*) \\ A_{i,z}H_{i,j,z,z-1} - B_{i,z}F_{i,j,z,z-1} & -P_{j,l,z} \end{pmatrix} < 0$$

When using the Lyapunov function (6.15) we obtain

Corollary 6.8 *The switching TS system* (6.1) *is asymptotically stabilized by the switching control law* (6.12) *if there exist matrices* $H_{i,j,m,n}$ *and symmetric positive definite matrices* $P_{i,j,m,n} = P_{i,j,m,n}^T > 0,\ i,j = 1,2,\ldots,n_s,\ (v_i,v_j) \in \mathscr{E},\ (v_j,v_l) \in \mathscr{E},\ m,n = 1,2,\ldots,r_i$, *so that*

$$\begin{pmatrix} -P_{i,j,z,z-1} & (*) \\ A_{i,z}H_{i,j,z,z-1} - B_{i,z}F_{i,j,z,z-1} & -H_{j,l,z,z+1} + (*) + P_{j,l,z,z+1} \end{pmatrix} < 0$$

Note that similarly to Corollaries 6.5 and 6.6, the two results above are not equivalent. In order to further reduce the conservativeness of the results, the two extensions above can also be combined, i.e., one may use the control law defined on switches with delayed Lyapunov functions.

6.3 Stabilization by switching

In this section, we consider the problem of determining a switching law which stabilizes a given switching system of the form

$$\boldsymbol{x}(k+1) = A_{j,z}\boldsymbol{x}(k) \tag{6.16}$$

While it is possible to also consider a control input, to illustrate the idea, we consider the system (6.16). Since we do not have a control input, the only way to stabilize this system is by a suitable choice of the switches.

The following assumptions are made:

1. a stable subsystem cannot be active for an infinite number of samples
2. the graph associated to the system is strongly connected: from any initial subsystem there exists a path to any other subsystem

Let us discuss the assumptions above. Recall that our goal is to obtain convergence of all the states to zero. Since there is no control input, we have to do this by either keeping continuously active a subsystem that is stable or by switching between the subsystems.

Assumption 1 is related to the existence of stable subsystems. If there exists a stable subsystem that can be active for an infinite number of samples, the states converge to zero if this subsystem is kept active. If the initial condition implies a different subsystem, stabilization is obtained by switching to the stable subsystem and keeping it active. Thus, on the level of the whole switching system, the problem can be reformulated as finding a path from each subsystem to this stable one. If such a path exists, the problem is solved. However, if no stable subsystem can be continuously active, a law that continuously switches between subsystems has to be constructed to obtain convergence of the states to zero.

Assumption 2 is related to the fact that the switching law may not contain all the subsystems. However, if the associated graph is strongly connected, meaning that there exists a path from any subsystem to any other subsystem, it is possible to switch from any initial subsystem to one that is contained in the switching law. The case when Assumption 2 is not satisfied will be discussed later on.

6.3.1 Stabilizing switching law

Recall that we consider the associated directed graph $\mathscr{G} = \{\mathscr{V}, \mathscr{E}\}$, where the vertices $\mathscr{V} = \{v_1, v_2, \ldots, v_{n_s}\}$ correspond to the subsystems, and each edge $e_{i,j} = (v_i, v_j) \in \mathscr{E}$ corresponds to an admissible transition.

In what follows, we build a weight-adjacency matrix, that assigns to each admissible transition, including self-transitions, a weight. By convention, if $(v_i, v_j) \notin \mathscr{E}$, for $i \neq j$, $i, j = 1, 2, \ldots, n_s$ the corresponding weight $w_{i,j} = \infty$, and if $(v_i, v_i) \notin \mathscr{E}$, $i = 1, 2, \ldots, n_s$, then the corresponding weight $w_{i,i} = 1$. For all other edges, the weight will be determined by the properties of the corresponding switch.

For this, consider the Lyapunov function

$$V(\boldsymbol{x}(k)) = \boldsymbol{x}(k)^T P_{i,z} \boldsymbol{x}(k) \tag{6.17}$$

with $P_{i,z} = P_{i,z}^T > 0$, for the ith subsystem, $i = 1, 2, \ldots, n_s$.

Using these Lyapunov functions our goal is to determine upper bounds on their increase for each switch, i.e., finding constants $\delta_{i,j} > 0$ so that $V(\boldsymbol{x}(k+1)) \leq \delta_{i,j} V(\boldsymbol{x}(k))$ for the transition (v_i, v_j).

For any $(v_i, v_j) \in \mathscr{E}$ we have

$$\begin{aligned}
&V(\boldsymbol{x}(k+1)) - \delta_{i,j} V(\boldsymbol{x}(k)) < 0\\
&\boldsymbol{x}(k+1)^T P_{j,z+1} \boldsymbol{x}(k+1) - \delta_{i,j} \boldsymbol{x}(k)^T P_{i,z} \boldsymbol{x}(k) < 0\\
&\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -\delta_{i,j} P_{i,z} & 0 \\ 0 & P_{j,z+1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} < 0
\end{aligned}$$

At the same time, the system's dynamics can be written as

$$\begin{pmatrix} A_{i,z} & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} = 0$$

Using Lemma 2.13, we have $V(\boldsymbol{x}(k+1)) - \delta_{i,j}V(\boldsymbol{x}(k)) < 0$ if there exists M so that

$$\begin{pmatrix} -\delta_{i,j}P_{i,z} & 0 \\ 0 & P_{j,z+1} \end{pmatrix} + M \begin{pmatrix} A_{i,z} & -I \end{pmatrix} + (*) < 0$$

Choosing $M = \begin{pmatrix} 0 \\ M_{i,j,z} \end{pmatrix}$ we obtain the sufficient conditions

$$\begin{pmatrix} -\delta_{i,j}P_{i,z} & (*) \\ M_{i,j,z}A_{i,z} & -M_{i,j,z} - M_{i,j,z}^T + P_{j,z+1} \end{pmatrix} < 0 \tag{6.18}$$

To find all $\delta_{i,j}$, one has to solve (6.18) for all $(v_i, v_j) \in \mathscr{E}$.

Once all $\delta_{i,j}$ are available, define the weight matrix as $\mathscr{W} = [w_{i,j}]$ with

$$w_{i,j} = \begin{cases} \delta_{i,i}^{p_i^m - 1} & \text{if } \delta_{i,i} > 1 \\ \delta_{i,i}^{p_i^M - 1} & \text{if } \delta_{i,i} < 1 \\ \delta_{i,j} & \text{if } i \neq j \end{cases} \tag{6.19}$$

Then, the following result can be stated:

Theorem 6.9 *The switching system* (6.16) *is asymptotically stabilizable by a switching law, if its associated graph $\mathscr{G} = \{\mathscr{V}, \mathscr{E}, \mathscr{W}\}$ contains a subunitary cycle $\mathscr{C}_n$. Furthermore, for the ith initial subsystem, $i = 1, 2, \ldots, n_s$, the stabilizing switching law is given by $\mathscr{P}(v_i, v_j)\mathscr{C}_n$, where v_i denotes the vertex corresponding to the initial subsystems, $\mathscr{P}(v_i, v_j)$ is a path to vertex v_j, with $v_j \in \mathscr{C}_n$.*

Proof: Consider the switching system (6.16) and recall that the associated graph description is $\mathscr{G} = \{\mathscr{V}, \mathscr{E}, \mathscr{W}\}$, where the vertices $\mathscr{V} = \{v_1, v_2, \ldots, v_{n_s}\}$ correspond to the subsystems, each edge $e_{i,j} = (v_i, v_j) \in \mathscr{E}$ corresponds to an admissible transition between two subsystems, and the weight matrix $\mathscr{W}$ is the one constructed above.

Assume that there exists a subsunitary cycle, i.e., there exists $\mathscr{C}_n = \{v_{c1}, v_{c2}, \ldots, v_{cp}, v_{c1}\}$ such that the product of the edges and nodes in this cycle is subunitary, and let this product be denoted by δ_n. For any subsystem i such that $v_i \in \mathscr{C}_n$, we have $V(\boldsymbol{x}(k+n_c)) < \delta_n V(\boldsymbol{x}(k))$, i.e., after a full cycle, the corresponding Lyapunov function increases at most δ_n times, with $\delta_n < 1$. Consequently, the periodic switching law $\mathscr{C}_n = [c_1, c_2, \ldots, c_p, c_1]$ stabilizes the periodic system given by this cycle.

Let us now see the subsystems (if any) that are not in $\mathscr{C}_n$. Since we assumed that the $\mathscr{G}$ is strongly connected (Assumption 2), for all $v_i \notin \mathscr{C}$, there exists a path $\mathscr{P}(v_i, v_j)$ from the ith subsystem to a subsystem j on the cycle. Consider the switching law $\mathscr{P}(v_i, v_j)\mathscr{C}_n$, i.e., first a switching law that leads to the cycle and then the periodic switching law corresponding to the cycle. Since none of the subsystems have a finite escape time, even though during the switches corresponding to $\mathscr{P}(v_i, v_j)$ the

Lyapunov function might increase, during the periodic switching it will eventually decrease and consequently stabilize the system. ■

It should be noted that, if it exists, the periodic stabilizing law may not be unique. Let us construct the weight matrix as $\mathscr{W} = [w_{i,j}]$ with

$$w_{i,j} = \begin{cases} \delta_{i,i}^{q_i} & \text{if } i = j \\ \delta_{i,j} & \text{if } i \neq j \end{cases}$$

with $q_i \in \{p_i^m - 1, p_i^m, \ldots, p_i^M - 1\}$. If the associated graph $\mathscr{G} = \{\mathscr{V}, \mathscr{E}, \mathscr{W}\}$ contains a subunitary cycle $\mathscr{C}_n$, then $\mathscr{C}_n$ will give a stabilizing periodic law. The proof is similar to that of Theorem 6.9.

A quantifiable difference in the stabilizing switching laws consists in the guaranteed lowest upper bound on the convergence rate obtained. With the weight matrix constructed as above, once the periodic law becomes active, the Lyapunov function decreases in each cycle δ_n times, with

$$\begin{aligned} \delta_n &= \prod_{k=1}^{n_p} w_{p_k,p_k} \cdot \prod_{k=1}^{n_p-1} w_{p_k,p_{k+1}} \\ &= \prod_{k=1}^{n_p} \delta_{v_{p_k},v_{p_k}}^{q_{v_{p_k}}} \cdot \prod_{k=1}^{n_p-1} \delta_{v_{p_k},v_{p_{k+1}}} \end{aligned}$$

where $(v_{p_k}, v_{p_{k+1}}) \in \mathscr{C}_n$. Since each element in the product is positive, the product is minimized by choosing the minimal admissible $q_{v_{p_k}}$ if $\delta_{v_{p_k},v_{p_k}} > 1$ and the maximal admissible $q_{v_{p_k}}$ if $\delta_{v_{p_k},v_{p_k}} < 1$. Thus, by choosing the weight matrix (6.19), we obtain the periodic law with the lowest upper bound on the convergence rate.

It has to be noted that we use the one-sum Lyapunov function $V = \boldsymbol{x}(k)^T P_{i,z} \boldsymbol{x}(k)$ for the easier notation and derivation. The conditions (6.18) can be easily derived for the n-sum Lyapunov function $V = \boldsymbol{x}(k)^T P_{i,\underbrace{z\ldots z}_{n}} \boldsymbol{x}(k)$ and using the n-sum matrix $M_{i,j,\underbrace{z\ldots z}_{n}}$, leading to (Megretski, 1996)

$$\begin{pmatrix} -\delta_{i,j} P_{i,\underbrace{z\ldots z}_{n}} & (*) \\ M_{i,j,\underbrace{z\ldots z}_{n}} A_{i,z} & -M_{i,j,\underbrace{z\ldots z}_{n}} - M_{i,j,\underbrace{z\ldots z}_{n}}^T + P_{j,\underbrace{z+1\ldots z+1}_{n}} \end{pmatrix} < 0$$

On the other hand, conditions to determine that there is no switching law that can stabilize a given system can also be obtained. Consider the Lyapunov function (6.17) and let us now find lower bounds on the increase of the Lyapunov function in one sample, i.e., constants $\omega_{i,j} > 0$ so that $V(\boldsymbol{x}(k+1)) \geq \omega_{i,j} V(\boldsymbol{x}(k))$, if the transition is (v_i, v_j).

For any $(v_i, v_j) \in \mathscr{E}$ we have

$$
\begin{aligned}
&V(\boldsymbol{x}(k+1)) - \omega_{i,j}V(\boldsymbol{x}(k)) > 0\\
&\boldsymbol{x}(k+1)^T P_{j,z+1}\boldsymbol{x}(k+1) - \omega_{i,j}\boldsymbol{x}(k)^T P_{i,z}\boldsymbol{x}(k) > 0\\
&\begin{pmatrix}\boldsymbol{x}(k)\\ \boldsymbol{x}(k+1)\end{pmatrix}^T \begin{pmatrix}-\omega_{i,j}P_{i,z} & 0\\ 0 & P_{j,z+1}\end{pmatrix}\begin{pmatrix}\boldsymbol{x}(k)\\ \boldsymbol{x}(k+1)\end{pmatrix} > 0
\end{aligned}
$$

Combining it with the system's dynamics and choosing $M = \begin{pmatrix}M_{i,j,z}\\ 0\end{pmatrix}$ in Finsler's lemma, we obtain the conditions

$$
\begin{pmatrix}-M_{i,j,z}A_{i,z} + (*) + \omega_{i,j}P_{i,z} & M_{i,j,z}\\ (*) & P_{j,z+1}\end{pmatrix} > 0 \tag{6.20}
$$

Now, define the lower weight matrix as $\underline{\mathscr{W}} = [\underline{w}_{i,j}]$ with

$$
\underline{w}_{i,j} = \begin{cases}\omega_{i,i}^{p_i^m} & \text{if } \omega_{i,i} > 1\\ \omega_{i,i}^{p_i^M} & \text{if } \omega_{i,i} < 1\\ \omega_{i,j} & \text{if } i \neq j\end{cases}
$$

and the following result can be stated:

Corollary 6.10 *The switching system* (6.16) *is not stabilizable by a switching law, if all the cycles in the associated graph* $\mathscr{G} = \{\mathscr{V}, \mathscr{E}, \underline{\mathscr{W}}\}$ *are supraunitary.*

6.3.2 Discussion and examples

The first issue that has to be noted is that due to the term δP, the conditions (6.18) are BMIs. Although the solution will not be optimal, however, these BMIs can be solved in several ways using two-step LMI conditions. A first possibility is

1. Solve (6.18) for all $i = j$, i.e., for each subsystem. In this case, the $P_{i,z}$s are independent, and each condition can be formulated as a *gevp* problem. For this step, we have the conditions

$$
\begin{pmatrix}-\delta_{i,i}P_{i,z} & (*)\\ M_{i,i,z}A_{i,z} & -M_{i,i,z} - M_{i,i,z}^T + P_{i,z+1}\end{pmatrix} < 0
$$

   for which sufficient conditions can be formulated using e.g., Lemma 2.11, with

$$
\Gamma_{j,k}^{i,l} = \begin{pmatrix}-\delta_{i,i}P_{i,j} & (*)\\ M_{i,i,j}A_{i,k} & -M_{i,i,j} - M_{i,i,j}^T + P_{i,l}\end{pmatrix}
$$

   for $i = 1, 2, \ldots, n_s$. For each subsystem, the *gevp* problem can be independently solved. Note that in the expression above i denotes the index of the subsystem and l corresponds to a different time instant.

2. Once the $P_{i,z}$s are obtained, using them one can solve the conditions for different subsystems, i.e,

$$\begin{pmatrix} -\delta_{i,j} P_{i,z} & (*) \\ M_{i,j,z} A_{i,z} & -M_{i,j,z} - M_{i,j,z}^T + P_{j,z+1} \end{pmatrix} < 0$$

with $i, j = 1, 2, \dots, n_s$, $i \neq j$, for the decision variables $\delta_{i,j}$ and $M_{i,j,z}$. Sufficient LMI conditions can again be formulated using e.g., Lemma 2.11.

Another possibility is solving all the conditions (6.18) first for the decision variables $P_{i,z}$ and $M_{i,j,z}$, $i, j = 1, 2, \dots, n_s$ and afterward for $\delta_{i,j}$ and $M_{i,j,z}$ $i, j = 1, 2, \dots, n_s$, minimizing at the same time $\delta_{i,j}$. Alternatively, one can use a path-following algorithm or available BMI solvers, such as *penbmi* (Kočvara and Stingl, 2008).

Since $\delta_{i,j}$, $i, j = 1, 2, \dots, n_s$, correspond to an upper bound on the increase or decrease of the Lyapunov function during a transition from the ith to the jth subsystem, the conditions (6.18) always have a solution with large enough $\delta_{i,j}$, $i, j = 1, 2, \dots, n_s$. A special case is when the transition is a self-transition, and the constant is subunitary, i.e., $\delta_{i,i} < 1$, in which case this constant corresponds to the decay-rate of the subsystem. We do not require that $\delta_{i,j}$, $i, j = 1, 2, \dots, n_s$, are all sub-unitary. Indeed, our goal is to find a subunitary cycle, for which it may be enough if there exists a single $\delta_{i,j} < 1$.

The problem of finding a subunitary cycle can be approached in two ways. On the one hand, one can first solve the conditions (6.18) for the whole graph and find the subunitary cycles. This can be done by using the logarithm of $\delta_{i,j}$ as weights in the graph and employing the methods proposed e.g., in (Cherkassky and Goldberg, 1999; Yamada and Kinoshita, 2002; Hanusa, 2009). On the other hand, one can generate all elementary cycles in the graph, and solve independently the conditions (6.18) for each cycle until a subunitary cycle is found. This approach will be illustrated in what follows.

Example 6.5 Consider the switching system composed of four subsystems, with the associated graph from Example 4.1, i.e.,

$$\boldsymbol{x}(k+1) = A_{i,z} \boldsymbol{x}(k)$$

for $i = 1, 2, 3, 4$, and with admissible switches $(1,2)$, $(2,1)$, $(2,3)$, $(3,1)$, $(4,3)$, $(1,4)$. The graph can be defined as $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, with $\mathcal{V} = \{1, 2, 3, 4\}$ and $\mathcal{E} = \{(1,1), (1,2), (1,4), (2,1), (2,2), (2,3), (3,1), (3,3), (4,3), (4,4)\}$.

The local matrices and membership functions are given as:

$$
\begin{aligned}
A_{1,1} &= \begin{pmatrix} 0.5 & 0.2 \\ 0.85 & 0.2 \end{pmatrix} & A_{1,2} &= \begin{pmatrix} 0.75 & 0.3 \\ 0.2 & 0.5 \end{pmatrix} \\
h_{1,1} &= \frac{1-\sin(x_1(k))}{2} & h_{1,2} &= 1-h_{1,1} \\
A_{2,1} &= \begin{pmatrix} 1.3 & 0.6 \\ -0.22 & 0.5 \end{pmatrix} & A_{2,2} &= \begin{pmatrix} 0.5 & -0.15 \\ -0.8 & 0.8 \end{pmatrix} \\
h_{2,1} &= \frac{1-\cos(x_1(k))}{2} & h_{2,2} &= 1-h_{2,1} \\
A_{3,1} &= \begin{pmatrix} 0.8 & 0.41 \\ 0.6 & 0.2 \end{pmatrix} & A_{3,2} &= \begin{pmatrix} 0.06 & 0.81 \\ 0.35 & 0.1 \end{pmatrix} \\
h_{3,1} &= 1-\exp(-x_1^2(k)) & h_{3,2} &= 1-h_{3,1} \\
A_{4,1} &= \begin{pmatrix} 0.45 & 0.4 \\ 0.8 & 0.4 \end{pmatrix} & A_{4,2} &= \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.6 \end{pmatrix} \\
h_{4,1} &= \frac{1-\tanh(x_1(k))}{2} & h_{4,2} &= 1-h_{4,1}
\end{aligned}
$$

where $x_1(k)$ denotes the first element of the state vector $\boldsymbol{x}$ at time k. Each subsystem can be active for at least 2 and at most 5 samples, i.e., $p_i^m = 2$ and $p_i^M = 5$, $i = 1, 2, 3, 4$. Our goal is to find a switching law that can stabilize this system.

The first subsystem is stable. However, this subsystem can only be active for at most 5 samples. The remaining subsystems are not stable: both local models of the second subsystem, the first local model of the third and the second local model of the fourth subsystem have eigenvalues larger than one.

Let us now inspect the associated graph of admissible switches. Although the number of switches is quite large, there are only three elementary cycles: $\mathscr{C}_1 = [1, 2, 1]$, $\mathscr{C}_2 = [1, 2, 3, 1]$, and $\mathscr{C}_3 = [1, 4, 3, 1]$. All the remaining cycles are equivalent to one of these three. To ease the computations, we consider just the weights for the three enumerated cycles. We solve the conditions (6.18) using the two-step procedure explained above[1], i.e., first solving the *gevp* problems associated to each subsystem and then the interconnection LMIs. We obtain:

1. Cycle $\mathscr{C}_1 = [1, 2, 1]$:

$$
\delta = \begin{pmatrix} 0.8333 & 1.6743 & \times & \times \\ 2.0096 & 1.3564 & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}
$$

[1]Similar results are obtained using *penbmi* (Kočvara and Stingl, 2008).

The weight matrix[2] is

$$\mathscr{W}_1 = \begin{pmatrix} 0.4822 & 1.6743 & \times & \times \\ 2.0096 & 1.3564 & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}$$

The weight of the cycle $W(\mathscr{C}_1) = 2.20 > 1$, consequently no conclusion can be drawn.

2. Cycle $\mathscr{C}_2 = [1, 2, 3, 1]$:

$$\delta = \begin{pmatrix} 0.8333 & 3.1171 & \times & \times \\ \times & 1.3564 & 1.5379 & \times \\ 0.5840 & \times & 1.1657 & \times \\ \times & \times & \times & \times \end{pmatrix}$$

The weight matrix is

$$\mathscr{W}_2 = \begin{pmatrix} 0.4822 & 3.1171 & \times & \times \\ \times & 1.3564 & 1.5379 & \times \\ 0.5840 & \times & 1.1657 & \times \\ \times & \times & \times & \times \end{pmatrix}$$

The weight of the cycle $W(\mathscr{C}_2) = 2.13 > 1$, consequently no conclusion can be drawn.

3. Cycle $\mathscr{C}_3 = [1, 4, 3, 1]$:

$$\delta = \begin{pmatrix} 0.8333 & \times & \times & 6.0713 \\ \times & 1.3564 & \times & \times \\ 0.3544 & \times & 1.1657 & \times \\ \times & \times & 0.6673 & 1.0643 \end{pmatrix}$$

The weight matrix is

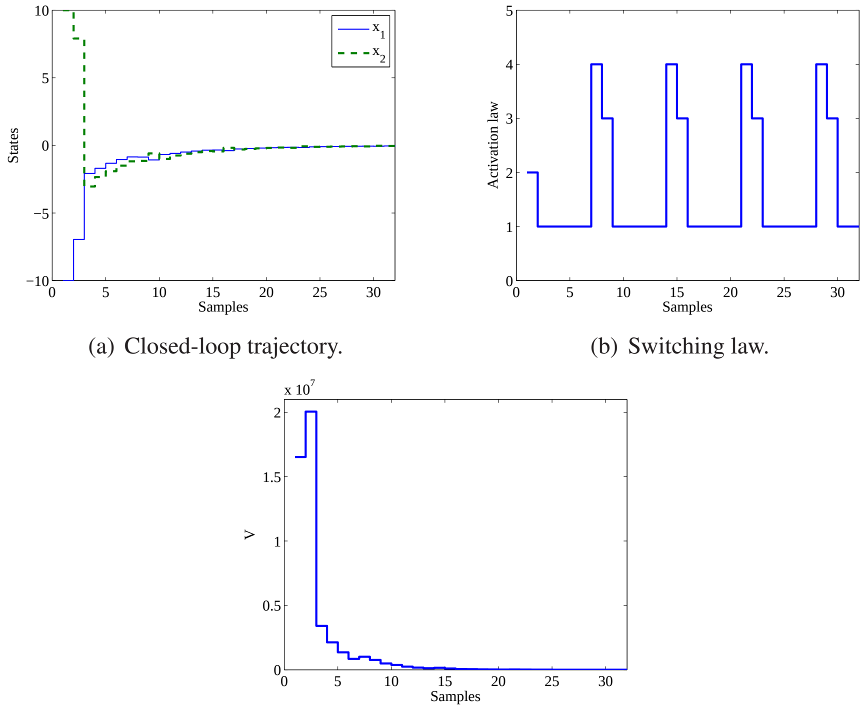
$$\mathscr{W}_3 = \begin{pmatrix} 0.4822 & \times & \times & 6.0713 \\ \times & 1.3564 & \times & \times \\ 0.3544 & \times & 1.1657 & \times \\ \times & \times & 0.6673 & 1.0643 \end{pmatrix}$$

The weight of the cycle $W(\mathscr{C}_3) = 0.8590 < 1$, therefore we have the stabilizing periodic law $[1, 1, 1, 1, 1, 4, 4, 3, 3, 1, \ldots]$.

[2] $\times$ denotes an element that is not of interest for the particular case.

The only subsystem that is not part of the periodic switching law that has been found is subsystem 2. However, if this is the initial subsystem, it can be stabilized by switching to any other subsystem (e.g., the first one) and then applying the periodic law. Such a trajectory is illustrated in Figure 6.5(a). The initial states were $\boldsymbol{x}_0 = [-10\,10]^T$, the initially active subsystem the second, and the trajectory has been obtained by first switching to the first subsystem and then applying the periodic law. As can be seen, the states converge to zero. The switching law is illustrated in Figure 6.5(b), and the value of the Lyapunov function in Figure 6.5(c). Although the value of the Lyapunov function increases for some switches, it decreases during a cycle.

Note that the periodic laws $[1, 1, 1, 1, 1, 4, 4, 4, 3, 3, 1, \ldots]$ and $[1, 1, 1, 1, 1, 4, 4, 4, 4, 3, 3, 1, \ldots]$ also stabilize the system. However, in this case the weights of the corresponding cycles are 0.9143 and 0.9731, respectively, thus, when applying one of these laws, the states will converge slower to zero.



(a) Closed-loop trajectory.

(b) Switching law.

(c) Value of the Lyapunov function

Figure 6.5: Simulation results for Example 6.5.

□

Let us now return to the assumption that the graph associated to the switching system should be strongly connected. Indeed, if this graph is strongly connected and if there exists a subunitary cycle, then starting from any subsystem, the states can be stabilized to zero. Let us assume now that the graph is not strongly connected. In this case, the existence of a stabilizing switching law depends on the initial subsystem. We illustrate this on an example.

Example 6.6 Consider a switching system composed of seven subsystems, with the associated graph illustrated in Figure 6.6.

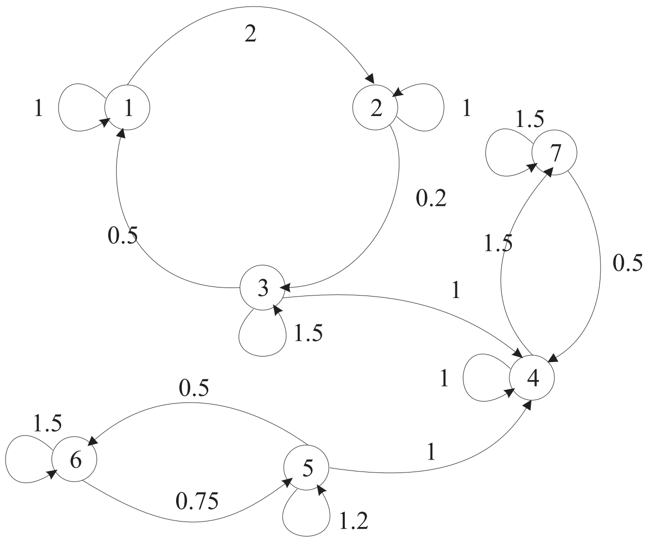


Figure 6.6: Graph representation of the switching system in Example 6.6.

Let us assume that after solving the BMIs in (6.18), the associated weight matrix is given by

$$\mathscr{W} = \begin{pmatrix} 1 & 2 & \infty & \infty & \infty & \infty & \infty \\ \infty & 1 & 0.2 & \infty & \infty & \infty & \infty \\ 0.5 & \infty & 1.5 & 2 & \infty & \infty & \infty \\ \infty & \infty & \infty & 1 & \infty & \infty & 1.5 \\ \infty & \infty & \infty & 1 & 1.2 & 0.5 & \infty \\ \infty & \infty & \infty & \infty & 0.75 & 1.5 & \infty \\ \infty & \infty & \infty & 0.5 & \infty & \infty & 1.5 \end{pmatrix}$$

where ∞ corresponds to an inadmissible switch. It can easily be seen from Figure 6.6 that there are two stable periodic switching laws: one involving subsystems 1, 2, and 3, with weight $W_{1,2,3} = 1 \cdot 2 \cdot 1 \cdot 0.2 \cdot 1.5 \cdot 0.5 = 0.3$ and a second involving subsystems 5 and 6 with weight $W_{5,6} = 1.2 \cdot 0.5 \cdot 1.5 \cdot 0.75 = 0.675$. Consequently, starting from subsystems 1, 2, 3, 5, or 6, the switching system can be stabilized, by choosing the corresponding periodic switching law. Let us inspect now subsystems 4 and 7. From these subsystems there are no paths to any other subsystems, and the weight of the cycle $W(\mathscr{C}_{4,7}) = 1.125 > 1$, therefore no conclusion can be drawn. □

To summarize, our goal has been to stabilize a switching system, i.e., obtain convergence of all the states to zero. The only instrument to do this – since there is no control input – is by switching between the subsystems. We construct a periodic switching law that stabilizes the system. This switching law may not contain all the subsystems. If the subsystems are strongly connected, even if a subsystem is not part of the switching law – such as subsystem 2 in Example 4.1, which is only activated if it is the starting subsystem – there exists a path to switch to a subsystem that is part of the switching law. If the graph is not strongly connected, we might not be able to switch to a subsystem that is "on" the switching law, and therefore each strongly connected subgraph has to be treated separately. Such a case is discussed in Example 6.6: starting from subsystems 4 or 7, we cannot switch to a subsystem contained in a stabilizing switching law and thus, for such initial conditions, the system may not be stabilized.

To compare our results to recent ones from the literature, we adapt Example 1 from (Zhang et al., 2009).

Example 6.7 Consider the switching system with two linear discrete-time subsystems with the system matrices (Zhang et al., 2009)

$$A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1.5 & 1 \\ 0 & 1.5 \end{pmatrix}$$
$$B_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

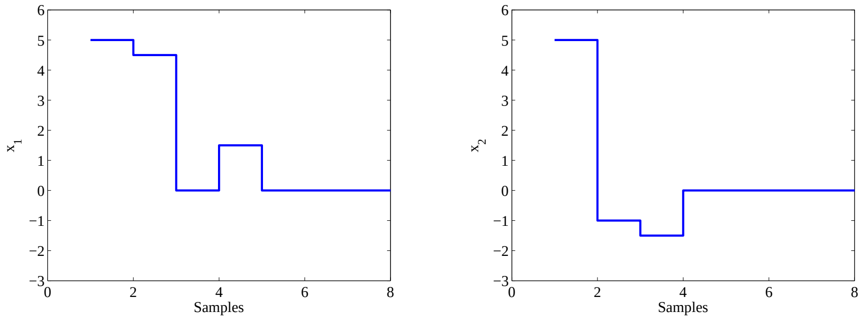
Since in the present approach we do not consider controller gain design, the control gains have been designed separately and have the values

$$K_1 = \begin{pmatrix} 0.1 & 1 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 1.5 & 1 \end{pmatrix}$$

Thus, the closed-loop system matrices are (with a slight abuse of notation)

$$A_1 = \begin{pmatrix} 1.9 & -1 \\ -0.2 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1.5 \end{pmatrix}$$

It can easily be verified that none of the matrices is Schur. However, by simply switching between the subsystems, the states will converge to zero, thus a stabilizing switching law is $[1, 2, 1, 2, \ldots]$. The state trajectories of the closed-loop system, starting from $\boldsymbol{x} = [5, 5]^T$ and using this switching law, are illustrated in Figures 6.7(a) and 6.7(b).



(a) Trajectory of x_1. (b) Trajectory of x_2.

Figure 6.7: Simulation results for Example 6.7.

Let us now extend this system to a TS one, with local matrices

$$A_{1,1} = \begin{pmatrix} 1.9 & -1 \\ -0.2 & 0 \end{pmatrix} \qquad A_{1,2} = \begin{pmatrix} 1.9 & -.8 \\ -0.2 & 0 \end{pmatrix}$$
$$A_{2,1} = \begin{pmatrix} 0 & 0 \\ 0 & 1.5 \end{pmatrix} \qquad A_{2,2} = \begin{pmatrix} 0 & 0.5 \\ 0 & 1.5 \end{pmatrix}$$

Note that this system may represent a buck-boost power converter with uncertainty in the inductance, which appears in many practical problems.

For this system, not being linear, the results in (Zhang et al., 2009) no longer apply. However, following our approach, we have

$$\delta = \begin{pmatrix} 4.02 & 0.27 \\ 0.27 & 2.25 \end{pmatrix}$$

and the weight of the cycle $[1, 2, 1]$ is $4.02 \cdot 0.27 \cdot 2.25 \cdot 0.27 = 0.65 < 1$, thus a stabilizing switching law is $[1, 2, 1, 2, \ldots]$. □

6.4 Conclusions

In this chapter we considered stabilization of periodic and switching systems. First, we have developed conditions for the stabilization of periodic TS systems. We considered a periodic controller and the use of a periodic Lyapunov function, defined in the points where the subsystems switch. Using the developed conditions, one is able to design a controller for a system where the local models are not stable and not controllable.

Second, we have considered stabilization of switching – not necessarily periodic – TS systems. To develop the conditions, two switching Lyapunov functions have been used, leading to two sets of conditions. Their application has been illustrated on numerical examples. Extensions of these conditions using delays in the Lyapunov functions and the controller gains, in order to reduce their conservativeness, have also been presented.

Finally, we have investigated stabilization of switching nonlinear systems by a suitably chosen switching law. For this, first conditions that, when satisfied, guarantee that the system is stabilizable by switching law have been presented. For the development of the conditions, a switching Lyapunov function has been employed. If the conditions are satisfied, a switching law that stabilizes all the states to zero can be constructed.

All the presented conditions and switching laws have been illustrated on numerical examples.

The developed conditions are still conservative and there are many ways to relax them. For instance, one can use the nonquadratic Lyapunov functions and controller gains presented in Chapter 3 or employ any relaxation scheme from the literature. Another possible relaxation is the use of α-sample variation instead of a single one. Moreover, extensions to robust and H_∞ control are also possible. Furthermore, one may also consider the case when both controller gains and a switching law need to be designed, thus combining the results in Sections 6.2 and 6.3.

Chapter 7

Observer design

This chapter considers observer design of the periodic or switching system (4.3), repeated here for convenience:

$$\begin{aligned} \boldsymbol{x}(k+1) &= A_{j,z}\boldsymbol{x}(k) + B_{j,z}\boldsymbol{u}(k) \\ \boldsymbol{y}(k) &= C_{j,z}\boldsymbol{x}(k) \end{aligned} \tag{7.1}$$

We assume that the scheduling variables do not depend on the states that have to be estimated, and consequently they can be used in the observer.

First, we present conditions for designing an observer for the *periodic* system (7.1). Second, we show conditions for observer design such that the states of the *switching* system (7.1) – when the switches are not periodic and they cannot be influences – are asymptotically estimated. Finally, we present conditions for observer design under the assumption that the switching law can also be chosen. In all the cases the conditions are discussed and illustrated on numerical examples.

7.1 Observers for periodic systems

In this section, consider the observer design problem for the periodic TS model (7.1) with n_s subsystems, each subsystem j, $j = 1, 2, \ldots, n_s$, being active for p_j time samples. The observer we use is of the form

$$\begin{aligned} \boldsymbol{x}(k+1) &= A_{j,z}\widehat{\boldsymbol{x}}(k) + B_{j,z}\boldsymbol{u}(k) + H_{j,z}^{-1}L_{j,z}(\boldsymbol{y} - \widehat{\boldsymbol{y}}) \\ \widehat{\boldsymbol{y}}(k) &= C_{j,z}\widehat{\boldsymbol{x}}(k) \end{aligned} \tag{7.2}$$

that is also periodic, with the same periods as (7.1).

The estimation error is given by

$$\boldsymbol{e}(k+1) = (A_{j,z} - H_{j,z}^{-1}L_{j,z}C_{j,z})\boldsymbol{e}(k) \tag{7.3}$$

which is also a periodic system. The observer design conditions are equivalent to finding $H_{j,i}$ and $L_{j,i}$, $j = 1, 2, \ldots, n_s$, $i = 1, 2, \ldots, r_j$ so that (7.3) is asymptotically stable.

7.1.1 Design conditions

First, we consider observer design such that (7.3) is asymptotically stable. The following results can be stated.

Theorem 7.1 *The estimation error* (7.3) *is asymptotically stable, if there exist* $P_{j,i} = P_{j,i}^T > 0$, $H_{j,i}$, $L_{j,i}$ $j = 1, 2, \ldots, n_s$, $i = 1, 2, \ldots, r_j$, *such that the following conditions are satisfied:*

$$\begin{pmatrix} -P_{j,z} & (*) & \ldots & (*) & (*) \\ \Omega_{0,a} & \Omega_{0,b} & \ldots & (*) & (*) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \Omega_{p_{\underline{j+1}},a} & \Omega_{p_{\underline{j+1}},b} + P_{\underline{j+1},z+p_{\underline{j+1}}} \end{pmatrix} < 0 \tag{7.4}$$

where

$$\Omega_{l,a} = H_{\underline{j+1},z+l} A_{\underline{j+1},z+l} - L_{\underline{j+1},z+l} C_{\underline{j+1},z+l}$$
$$\Omega_{l,b} = -H_{\underline{j+1},z+l} + (*)$$

for $l = 0, \ldots, p_{\underline{j+1}} - 1$, *where* $\underline{j}$ *denotes the modulo of* j.

Remark: Note that $\underline{j+1}$ is used because due to the periodicity the $n_s + i$th subsystem is in fact the ith one.

Proof: Consider the following switching Lyapunov function, similar to the one used by Daafouz et al. (2002), but *defined only in the instants when a switching takes place* in the error dynamics:

$$V = \boldsymbol{e}(k)^T P_{j,z} \boldsymbol{e}(k)$$

for $j = 1, 2, \ldots, n_s$, if the active subsystem before the kth time instant was j.

The difference in the Lyapunov function is

$$\begin{aligned} &V(\boldsymbol{e}(k + p_{\underline{j+1}})) - V(\boldsymbol{e}(k)) = \\ &= \begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k + p_{\underline{j+1}}) \end{pmatrix}^T \begin{pmatrix} -P_{j,z} & 0 \\ 0 & P_{\underline{j+1},z+p_{\underline{j+1}}} \end{pmatrix} \begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k + p_{\underline{j+1}}) \end{pmatrix} \end{aligned}$$

The error dynamics during the $p_{\underline{j+1}}$ samples are

$$\Upsilon_{j+1} \begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k+1) \\ \vdots \\ \boldsymbol{e}(k + p_{\underline{j+1}}) \end{pmatrix} = 0$$

with

$$\Upsilon_{j+1} = \begin{pmatrix} G_{j+1,0} & -I & \dots & 0 & 0 \\ 0 & G_{j+1,1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & G_{j+1,p_{\underline{j+1}}-1} & -I \end{pmatrix}$$

with $G_{j+1,l} = A_{\underline{j+1},z+l} - H^{-1}_{\underline{j+1},z+l} L_{\underline{j+1},z+l} C_{\underline{j+1},z+l}$, $l = 0, 1, \dots, p_{\underline{j+1}} - 1$.

Using Lemma 2.13, the difference in the Lyapunov function is negative definite, if there exists M such that

$$\begin{pmatrix} -P_{j,z} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & P_{\underline{j+1},p_{\underline{j+1}}} \end{pmatrix} + M\Upsilon_{j+1} + (*) < 0$$

Choosing

$$M = \begin{pmatrix} 0 & 0 & \dots & 0 \\ H_{\underline{j+1},z} & 0 & \dots & 0 \\ 0 & H_{\underline{j+1},z+1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & H_{\underline{j+1},z+p_{\underline{j+1}}-1} \end{pmatrix}$$

leads directly to

$$\begin{pmatrix} -P_{j,z} & (*) & \dots & (*) & (*) \\ \Omega_{0,a} & \Omega_{0,b} & \dots & (*) & (*) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \Omega_{p_{\underline{j+1}},a} & \Omega_{p_{\underline{j+1}},b} + P_{\underline{j+1},z+p_{\underline{j+1}}} \end{pmatrix} < 0 \tag{7.5}$$

with

$$\Omega_{l,a} = H_{\underline{j+1},z+l} A_{\underline{j+1},z+l} - L_{\underline{j+1},z+l} C_{\underline{j+1},z+l}$$
$$\Omega_{l,b} = -H_{\underline{j+1},z+l} + (*)$$

for $l = 0, \dots, p_{\underline{j+1}} - 1$. ■

To reduce the conservativeness, in what follows, we extend the result above using an α-sample variation of the Lyapunov function. Then, the following conditions can be stated:

Theorem 7.2 *The periodic TS system* (7.3) *with periods* $p_1, p_2, \dots, p_{n_s}$ *is asymptotically stable, if there exist* $P_{ji} = P_{ji}^T > 0$, H_{ji}, L_{ji}, $j = 1, 2, \dots, n_s$, $i = 1, 2, \dots, r_j$,

$l = 1, 2, \ldots, \alpha$*, such that the following conditions are satisfied:*

$$\begin{pmatrix} -P_{j,z} & (*) & \ldots & (*) & (*) \\ \Omega_{j+1,0} & \bar{\Omega}_{j+1,0} & \ldots & (*) & (*) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \Omega_{j+\alpha,p_{\underline{j+\alpha}}-1} & \bar{\Omega}_{j+\alpha,t-1} + P_{\underline{j+\alpha},z+t} \end{pmatrix} < 0 \tag{7.6}$$

where $t = \sum_{i=1}^{\alpha} p_{\underline{j+i}}$*, and*

$$\begin{aligned} \Omega_{j+i,l} &= H_{\underline{j+i},z+l} A_{\underline{j+i},z+l} - L_{\underline{j+i},z+l} C_{\underline{j+i},z+l} \\ \bar{\Omega}_{j+i,l} &= -H_{\underline{j+i},z+l} + (*) \end{aligned}$$

for $l = 0, \ldots, t-1$*,* $i = 1, 2, \ldots, \alpha$*.*

Proof: Similarly to Theorem 7.1, consider the switching Lyapunov function defined only in the instants when a switching takes place in the error dynamics:

$$V = \boldsymbol{e}(k)^T P_{j,z} \boldsymbol{e}(k)$$

for $j = 1, 2, \ldots, n_s$, if the active subsystem was j.

Since the Lyapunov function is only defined in the switching instants, the α-difference in the Lyapunov function corresponds to α consecutive switches in the system. Consequently, the α-difference *in the Lyapunov function* is

$$\begin{aligned} &V(\boldsymbol{e}(k+t)) - V(\boldsymbol{e}(k)) = \\ &= \begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k+t) \end{pmatrix}^T \begin{pmatrix} -P_{j,z} & 0 \\ 0 & P_{\underline{j+\alpha},z+t} \end{pmatrix} \begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k+t) \end{pmatrix} \end{aligned}$$

where $t = \sum_{i=1}^{\alpha} p_{\underline{j+i}}$.

The error dynamics during the t samples corresponding to the α switches in the system are

$$\Gamma_{j+1:j+\alpha} \begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k+1) \\ \vdots \\ \boldsymbol{e}(k+t) \end{pmatrix} = 0$$

with

$$\Gamma_{j+1:j+\alpha} = \begin{pmatrix} G_{\underline{j+1},z} & -I & \ldots & 0 & 0 \\ 0 & G_{\underline{j+1},z+1} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & G_{\underline{j+\alpha},z+t-1} & -I \end{pmatrix}$$

with $G_{\underline{j+i},z+l} = A_{\underline{j+i},z+l} - H^{-1}_{\underline{j+i},z+l} L_{\underline{j+i},z+l} C_{\underline{j+i},z+l}$, $i = 1, 2, \ldots, \alpha$, $l = 1, 2, \ldots, t-1$.

Using Lemma 2.13, the difference in the Lyapunov function is negative definite, if there exists M such that

$$\begin{pmatrix} -P_{j,z} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & P_{\underline{j+1},z+t} \end{pmatrix} + M\Gamma_{j+1:j+\alpha} + (*) < 0$$

Choosing

$$M = \begin{pmatrix} 0 & 0 & \dots & 0 \\ H_{\underline{j+1},z} & 0 & \dots & 0 \\ 0 & H_{\underline{j+1},z+1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & H_{\underline{j+\alpha},z+t-1} \end{pmatrix}$$

leads directly to (7.6). ■

7.1.2 Discussion

First, let us discuss how exactly the conditions derived in Section 7.1.1 are applied. For simplicity, consider a switching TS model consisting of two subsystems, and without input, i.e., we have:

$$\begin{aligned} \boldsymbol{x}(k+1) &= \begin{cases} \sum_{i=1}^{r_1} h_{1i}(\boldsymbol{z}_1(k))A_{1i}\boldsymbol{x}(k) \\ \sum_{i=1}^{r_2} h_{2i}(\boldsymbol{z}_2(k))A_{2i}\boldsymbol{x}(k) \end{cases} \\ \boldsymbol{y}(k) &= \begin{cases} \sum_{i=1}^{r_1} h_{1i}(\boldsymbol{z}_1(k))C_1\boldsymbol{x}(k) \\ \sum_{i=1}^{r_2} h_{2i}(\boldsymbol{z}_2(k))C_2\boldsymbol{x}(k) \end{cases} \end{aligned} \tag{7.7}$$

Assume that the period of the first subsystem is 2, and the period of the second subsystem is 1, i.e., $p_1 = 2$ and $p_2 = 1$. The switching in the system and in the Lyapunov function are depicted in Figure 7.1. As can be seen, the Lyapunov function (with matrices P_1 and P_2) is defined only in the moments when there is a switching in the system: from $A_{1,z}$ to $A_{2,z}$ or from A_{2z} to A_{1z}, respectively. A 1-sample variation of the Lyapunov function corresponds to the difference between two consecutive values of the Lyapunov function. A 2-sample variation corresponds to the difference after 2 samples of the Lyapunov function, etc.

For system (7.7), the conditions of Theorem 7.1 correspond to *there exist* $P_{j,i} = P_{j,i}^T > 0$, $H_{j,i}$, $L_{j,i}$, $j = 1, 2$, $i = 1, 2, \dots, r_j$, *so that the following conditions are satis-*

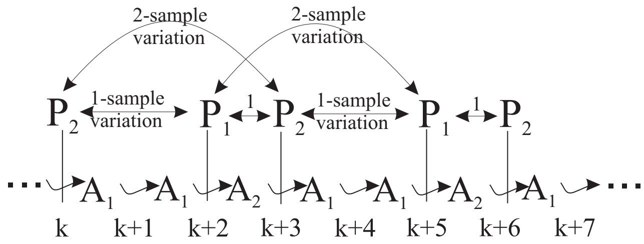


Figure 7.1: Switches in the system and in the Lyapunov function.

fied:

$$\begin{gathered}\begin{pmatrix}-P_{1,z} & (*)\\ \Omega_{2,0} & -H_{2,z}+(*)+P_{2,z+1}\end{pmatrix}<0\\ \begin{pmatrix}-P_{2,z} & (*) & (*)\\ \Omega_{1,0} & -H_{1,z}+(*) & (*)\\ 0 & \Omega_{1,1} & -H_{1,z+1}+(*)+P_{1,z+2}\end{pmatrix}<0\end{gathered} \tag{7.8}$$

with $\Omega_{i,l}=H_{i,z+l}A_{i,z+l}-L_{i,z+l}C_{i,z+l}$, $i=1,2$, $l=0,1$. Relaxed LMI conditions can be formulated using Lemmas 2.10 and 2.11, e.g.,

Corollary 7.3 *The system* (7.7) *is asymptotically stable if there exist* $P_{j,i}=P_{j,i}^T>0$, $H_{j,i}$, $j,i=1,2$, *so that*

$$\begin{aligned}&\Gamma^1_{i_1i_2i_3i_4i_5}+\Gamma^1_{i_2i_1i_3i_4i_5}+\Gamma^1_{i_1i_2i_4i_3i_5}+\Gamma^1_{i_2i_1i_4i_3i_5}<0\\&\Gamma^2_{i_1i_2i_3}+\Gamma^2_{i_2i_1i_3}<0\end{aligned}$$

$i_1,i_2,i_3,i_4,i_5=1,2$, *$i\leq j$, $m\leq n$, where*

$$\begin{aligned}\Gamma^1_{i_1i_2i_3i_4i_5}&=\begin{pmatrix}-P_{2,i_1} & (*) & (*)\\ \Omega_{1,i_1,i_2} & -H_{1,i_1}+(*) & (*)\\ 0 & \Omega_{1,i_3,i_4} & -H_{1,i_3}+(*)+P_{1,i_5}\end{pmatrix}\\ \Gamma^2_{i_1i_2i_3}&=\begin{pmatrix}-P_{1,i_1} & (*)\\ \Omega_{2,i_1,i_2} & -H_{2,i_1}+(*)+P_{2,i_3}\end{pmatrix}\end{aligned}$$

where $\Omega_{l+1,i_1,i_2}=H_{\underline{l+1},i_1}A_{\underline{l+1},i_2}-L_{\underline{l+1},i_1}C_{\underline{l+1},i_2}$, $l=1,2$, $i_1,i_2=1,2,\ldots,r_l$.

Let us now consider a 2-sample variation of the Lyapunov function. The conditions of Theorem 7.2 become *there exist* $P_{j,i}=P_{j,i}^T>0$, $H_{j,i}$, $L_{j,i}$, $j,l=1,2$, $i=$

$1, 2, \ldots, r_j$, *so that the following conditions are satisfied:*

$$\begin{pmatrix} -P_{1,z} & (*) & (*) & (*) \\ \Omega_{2,0} & \bar{\Omega}_{2,0} & (*) & (*) \\ 0 & \Omega_{2,1} & \bar{\Omega}_{2,1} & (*) \\ 0 & 0 & \Omega_{1,2} & \bar{\Omega}_{1,2} + P_{1,z+3} \end{pmatrix} < 0$$

$$\begin{pmatrix} -P_{2,z} & (*) & (*) & (*) \\ \Omega_{1,0} & \bar{\Omega}_{1,0} & (*) & (*) \\ 0 & \Omega_{1,1} & \bar{\Omega}_{1,1} & (*) \\ 0 & 0 & \Omega_{2,2} & \bar{\Omega}_{2,2} + P_{2,z+3} \end{pmatrix} < 0$$

with $\Omega_{i,l} = H_{i,z+l}A_{i,z+l} - L_{i,z+l}C_{i,z+l}$, $\bar{\Omega}_{i,l} = -H_{i,z+l} + (*)$, $i = 1, 2$, $l = 0, 1$.

Similarly to the 1-sample variation, relaxed LMI conditions can easily be formulated.

Note that the conditions do not require that the the local matrices of the TS system are either stable or observable. We illustrate this on the following example.

Example 7.1 Consider the switching fuzzy system with two subsystems, each having period 2, i.e., $p_1 = p_2 = 2$ as follows:

$$\boldsymbol{x}(k+1) = \begin{cases} \sum_{i=1}^{2} h_{1i}(\boldsymbol{z}_1(k))A_{1i}\boldsymbol{x}(k) + Bu \\ \sum_{i=1}^{2} h_{2i}(\boldsymbol{z}_2(k))A_{2i}\boldsymbol{x}(k) + Bu \end{cases}$$

$$\boldsymbol{y}(k) = \begin{cases} \sum_{i=1}^{2} h_{1i}(\boldsymbol{z}_1(k))C_1\boldsymbol{x}(k) \\ \sum_{i=1}^{2} h_{2i}(\boldsymbol{z}_2(k))C_2\boldsymbol{x}(k) \end{cases}$$

with

$$A_{11} = \begin{pmatrix} 0.80 & 0.22 \\ -0.09 & 0.32 \end{pmatrix} \qquad A_{12} = \begin{pmatrix} -0.82 & -0.44 \\ -1.25 & 0.33 \end{pmatrix}$$

$$A_{21} = \begin{pmatrix} 0.44 & 0.46 \\ 0.93 & 0.41 \end{pmatrix} \qquad A_{22} = \begin{pmatrix} 0.84 & 0.20 \\ 0.52 & 0.67 \end{pmatrix}$$

$$C_{11} = C_{12} = \begin{pmatrix} 0 & 0 \end{pmatrix} \qquad C_{21} = C_{22} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$$

The local models A_{11} and A_{12} are not observable, since the measurement matrices C_{11} and C_{12} are zero. Moreover, A_{12} is unstable, its eigenvalues being $\begin{pmatrix} -1.1834 & 0.6934 \end{pmatrix}$. The membership functions are as follows. h_{11} is randomly generated in [0, 1], $h_{12} = 1 - h_{11}$ and $h_{21} = cos(x_1)^2$, $h_{22} = 1 - h_{21}$.

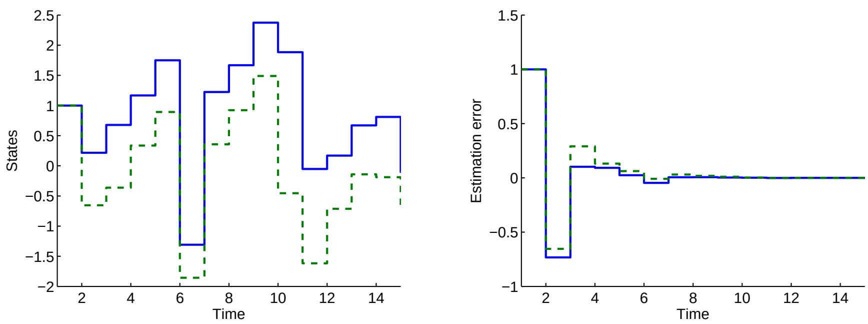
A trajectory of the states of the switching system, for the initial state $\begin{pmatrix} 1 & 1 \end{pmatrix}^T$ and a randomly generated input is illustrated in Figure 7.2(a).

Due to the unobservable and unstable local models, for this switching system it is not possible to design an observer using either quadratic or nonquadratic Lyapunov functions, that are common for both subsystems, as the LMIs available in the literature for observer design are unfeasible.

However, using the conditions of Theorem 7.1 we obtain a solution. The conditions used are those in (7.6). Solving them using the relaxation of Wang et al. (1996), we obtain

$$P_{11} = \begin{pmatrix} 0.55 & -0.21 \\ -0.21 & 0.54 \end{pmatrix} \qquad P_{12} = \begin{pmatrix} 0.55 & -0.22 \\ -0.22 & 0.56 \end{pmatrix}$$
$$H_{11} = \begin{pmatrix} 0.74 & -0.06 \\ -0.14 & 0.65 \end{pmatrix} \qquad H_{12} = \begin{pmatrix} 0.69 & -0.21 \\ -0.14 & 0.58 \end{pmatrix}$$
$$L_{11} = \begin{pmatrix} 0 & 0 \end{pmatrix}^T \qquad L_{12} = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$$
$$P_{21} = \begin{pmatrix} 1.04 & -0.08 \\ -0.08 & 0.71 \end{pmatrix} \qquad P_{22} = \begin{pmatrix} 1.17 & -0.08 \\ -0.08 & 0.76 \end{pmatrix}$$
$$H_{21} = \begin{pmatrix} 0.90 & -0.16 \\ 0.03 & 0.76 \end{pmatrix} \qquad H_{22} = \begin{pmatrix} 0.90 & 0.05 \\ -0.17 & 0.73 \end{pmatrix}$$
$$L_{21} = \begin{pmatrix} 0.36 & 0.84 \end{pmatrix} \qquad L_{22} = \begin{pmatrix} 0.87 & 0.40 \end{pmatrix}$$

A trajectory of the estimation error, with the estimated initial state being $\begin{pmatrix} 0 & 0 \end{pmatrix}^T$ is presented in Figure 7.2(b). As expected, the error converges to zero. □



(a) A trajectory of the states of the switching system.

(b) Estimation error using the designed observer.

Figure 7.2: Simulation results for Example 7.1.

7.2 Observer design for switching systems

In what follows, we present conditions to design a switching observer such that the estimation error dynamics converge to zero, with any admissible switching law. For this, we use a switching nonquadratic Lyapunov function and make use of its variation over possible switches. We assume that although the exact switching sequence is not known, the set of all the admissible switches is known. Furthermore, once a subsystem is activated, it will remain active for a number of samples, for which minimum and maximum bounds are known. Similarly to stability analysis and controller design, we use a directed graph representation of the switching system (7.1).

7.2.1 Design conditions

The switching observer is of the form (7.2) and the error dynamics, under the assumption that the scheduling variables are available online at sample k, are (repeated here for convenience)

$$\boldsymbol{e}(k+1) = (A_{j,z} - H_{j,z}^{-1} L_{j,z} C_{j,z})\boldsymbol{e}(k) \tag{7.9}$$

which in itself is a switching system.

To derive the observer design conditions, meaning that the error dynamics (7.9) should be asymptotically stable, consider the switching Lyapunov function

$$V = \boldsymbol{e}(k)^T P_{m,j,z} \boldsymbol{e}(k) \tag{7.10}$$

defined during the switches, i.e., on the edges of the associated graph $\mathscr{G} = \{\mathscr{V}, \mathscr{E}\}$, with $(v_m, v_j) \in \mathscr{E}$. If a subsystem j may be active for several number of samples, the edge (v_j, v_j) is also considered.

With this, the following result can be formulated:

Theorem 7.4 *The error dynamics* (7.9) *is asymptotically stable, if there exist* $P_{m,j,k} = P_{m,j,k}^T > 0$, $H_{j,k}$, $(v_m, v_j) \in \mathscr{E}$, $(v_j, v_l) \in \mathscr{E}$, $k = 1, 2, \ldots, r$, *such that*

$$\begin{pmatrix} -P_{m,j,z} & (*) \\ H_{j,z}A_{j,z} - L_{j,z}C_{j,z} & -H_{j,z} - H_{j,z}^T + P_{j,l,z+1} \end{pmatrix} < 0 \tag{7.11}$$

for all admissible paths $\mathscr{P}(v_m, v_l) = [v_m, v_j, v_l]$, $v_m \in \mathscr{V}$.

Proof: Consider the switching Lyapunov function (7.10), defined on the edges of the associated graph, with $\boldsymbol{e}(k)^T P_{m,j,z} \boldsymbol{e}(k)$ being active during the transition from vertex m to vertex j. The difference in the Lyapunov function for two consecutive samples is

$$\begin{aligned} \Delta V &= \boldsymbol{e}(k+1)^T P_{j,l,z+1} \boldsymbol{e}(k+1) - \boldsymbol{e}(k)^T P_{m,j,z} \boldsymbol{e}(k) \\ &= \begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k+1) \end{pmatrix}^T \begin{pmatrix} -P_{m,j,z} & 0 \\ 0 & P_{j,l,z+1} \end{pmatrix} \begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k+1) \end{pmatrix} \end{aligned}$$

where $[v_m, v_j, v_l]$ is an admissible path.

During the transition for j to l, the dynamics of the error system are described by

$$\begin{pmatrix} A_{j,z} - H_{j,z}^{-1} L_{j,z} C_{j,z} & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k+1) \end{pmatrix} = 0$$

Using Lemma 2.13, the difference in the Lyapunov function is negative, if there exists M such that

$$\begin{pmatrix} -P_{m,j,z} & 0 \\ 0 & P_{j,l,z+1} \end{pmatrix} + M \begin{pmatrix} A_{j,z} - H_{j,z}^{-1} L_{j,z} C_{j,z} & -I \end{pmatrix} + (*) < 0$$

By choosing

$$M = \begin{pmatrix} 0 \\ H_{j,z} \end{pmatrix}$$

we have directly (7.11). ■

Using Lemma 2.11, sufficient LMI conditions can be formulated, as follows:

Corollary 7.5 *The error dynamics* (7.9) *is asymptotically stable, if there exist* $P_{m,j,k} = P_{m,j,k}^T > 0$, $H_{j,k}$, $(v_m, v_j) \in \mathscr{E}$, $k, l = 1, 2, \ldots, r$, *such that*

$$\begin{aligned} &\Gamma_{kk}^{m,j,l,\gamma} < 0 \\ &\frac{2}{r-1}\Gamma_{kk}^{m,j,l,\gamma} + \Gamma_{k\beta}^{i,j,l,\gamma} + \Gamma_{\beta k}^{i,j,l,\gamma} < 0 \\ &k, \beta, \gamma = 1, 2, \ldots, r \end{aligned}$$

with

$$\Gamma_{k\beta}^{m,j,l,\gamma} = \begin{pmatrix} -P_{m,j,k} & (*) \\ H_{j,k} A_{j,\beta} - L_{j,k} C_{j,\beta} & -H_{j,k} + (*) + P_{j,l,\gamma} \end{pmatrix}$$

for all admissible paths $\mathscr{P}(v_m, v_l) = [v_m, v_j, v_l]$, $v_j \in \mathscr{V}$.

Remark: In order to reduce the conservativeness by exploiting the knowledge available of the switching sequence, one can also use the observer

$$\begin{aligned} \widehat{\boldsymbol{x}}(k+1) &= A_{j,z}\widehat{\boldsymbol{x}}(k) + B_{j,z}\boldsymbol{u}(k) + H_{m,j,z}^{-1} L_{m,j,z}(\boldsymbol{y}(k) - \widehat{\boldsymbol{y}}(k)) \\ \widehat{\boldsymbol{y}}(k) &= C_{j,z}\widehat{\boldsymbol{x}}(k) \end{aligned} \tag{7.12}$$

for the j-th subsystem, if the last switch has been from vertex m to vertex j.

The error dynamics $\boldsymbol{e}(k) = \boldsymbol{x}(k) - \widehat{\boldsymbol{x}}(k)$ using this observer can be written as

$$\begin{aligned} \boldsymbol{e}(k+1) &= A_{j,z}\boldsymbol{e}(k) - H_{m,j,z}^{-1} L_{m,j,z}(\boldsymbol{y}(k) - \widehat{\boldsymbol{y}}(k)) \\ &= (A_{j,z} - H_{m,j,z}^{-1} L_{m,j,z} C_{j,z})\boldsymbol{e}(k) \end{aligned} \tag{7.13}$$

Following the same steps as above, and in Finsler's lemma choosing

$$M = \begin{pmatrix} 0 \\ H_{m,j,z} \end{pmatrix}$$

we have

Corollary 7.6 *The error dynamics* (7.13) *is asymptotically stable, if there exist* $P_{m,j,k} = P_{m,j,k}^T > 0$, $H_{m,j,k}$, $(v_m, v_j) \in \mathscr{E}$, $(v_j, v_l) \in \mathscr{E}$, $k = 1, 2, \ldots, r$, *such that*

$$\begin{pmatrix} -P_{m,j,z} & (*) \\ H_{m,j,z}A_{j,z} - L_{m,j,z}C_{j,z} & -H_{m,j,z} + (*) + P_{j,l,z+1} \end{pmatrix} < 0 \tag{7.14}$$

for all admissible paths $\mathscr{P}(v_m, v_l) = [v_m, v_j, v_l]$, $v_m \in \mathscr{V}$.

The result above can be further extended to take into account more previous switches. However, this comes with the added computational cost and will eventually lead to considering all possible switching trajectories. To avoid this, but still reduce the conservativeness of the approach, let us now consider the α-sample variation of the Lyapunov function. Then, the following result can be formulated:

Theorem 7.7 *The error dynamics* (7.9) *is asymptotically stable, if there exist* $\alpha \in \mathbb{N}^+$, $P_{i,j,k} = P_{i,j,k}^T > 0$, $H_{j,k}$, $(v_i, v_j) \in \mathscr{E}$, $k = 1, 2, \ldots, r$, *such that*

$$\begin{pmatrix} -P_{v_0,v_1,z} & (*) & \ldots & (*) \\ \Omega_1 & -H_{v_1,z} + (*) & \ldots & (*) \\ 0 & \Omega_2 & \ldots & (*) \\ \vdots & \vdots & \ldots & -H_{v_\alpha,z+\alpha-1} + (*) + P_{v_\alpha,v_{\alpha+1},z+\alpha} \end{pmatrix} < 0 \tag{7.15}$$

for all admissible paths $\mathscr{P}(v_0, v_{\alpha+1}) = [v_0, v_1, \ldots, v_{\alpha+1}]$, *where* $\Omega_i = H_{v_i,z+i-1}A_{v_i,z+i-1} - L_{v_i,z+i-1}C_{v_i,z+i-1}$.

Proof: Consider the switching Lyapunov function (7.10), defined on the edges of the associated graph, with $P_{v_i,v_j,z}$ being active during the transition from vertex i to vertex j. The difference in the Lyapunov function for α consecutive samples is

$$\begin{aligned} \Delta V &= \boldsymbol{e}(k+\alpha)^T P_{v_\alpha,v_{\alpha+1},z+\alpha} \boldsymbol{e}(k+\alpha) - \boldsymbol{e}(k)^T P_{v_0,v_1,z} \boldsymbol{e}(k) \\ &= \begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k+\alpha) \end{pmatrix}^T \begin{pmatrix} -P_{v_0,v_1,z} & 0 \\ 0 & P_{v_\alpha,v_{\alpha+1},z+\alpha} \end{pmatrix} \begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k+\alpha) \end{pmatrix} \end{aligned}$$

where $[v_0, v_1, \ldots, v_{\alpha+1}]$ is an admissible path.

Along the switching sequence $[v_0, v_1, \ldots, v_{\alpha+1}]$, the error dynamics are described by

$$\begin{pmatrix} G_1 & -I & 0 & \ldots & 0 \\ 0 & G_2 & -I & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k+1) \\ \vdots \\ \boldsymbol{e}(k+\alpha) \end{pmatrix} = 0$$

with $G_i = A_{v_i,z+i-1} - H_{v_i,z+i-1}^{-1} L_{v_i,z+i-1} C_{v_i,z+i-1}$.

Following the same steps as in the proof of Theorem 7.4, using Lemma 2.13, the difference is the Lyapunov function is negative, if there exists M such that

$$\begin{pmatrix} -P_{v_0,v_1,z} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots \vdots & \\ 0 & 0 & \dots & P_{v_\alpha,v_{\alpha+1},z+\alpha} \end{pmatrix} + M \begin{pmatrix} G_1 & -I & 0 & \dots & 0 \\ 0 & G_2 & -I & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -I \end{pmatrix} + (*) < 0$$

with G_i defined as above.

By choosing

$$M = \begin{pmatrix} 0 & 0 & \dots & 0 \\ H_{v_1,z} & 0 & \dots & 0 \\ 0 & H_{v_2,z+1} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & H_{v_\alpha,z+\alpha-1} \end{pmatrix}$$

we have directly (7.15). ■

7.2.2 Example and discussion

Let us first illustrate the conditions of Theorem 7.4 on an example.

Example 7.2 Consider the switching system – actually a periodic switching system – illustrated in Figure 7.3. Assuming that none of the subsystems may be active

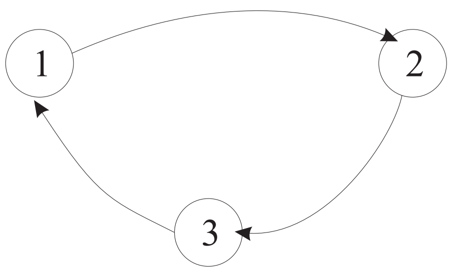


Figure 7.3: Periodic switching system for Example 7.2.

for more than one sample, i.e., $p^m = p^M = 1$, the graph is $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, with $\mathcal{V} = \{1, 2, 3\}$ and $\mathcal{E} = \{(1,2), (2,3), (3,1)\}$. Consider the following local models of the TS system above:

$$A_{1,1} = \begin{pmatrix} 0.61 & 0.10 \\ 1.82 & 0.50 \end{pmatrix} \qquad A_{1,2} = \begin{pmatrix} 0.21 & 1.59 \\ 0.34 & 0.77 \end{pmatrix}$$

$$C_{1,1} = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad C_{1,2} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$A_{2,1} = A_{2,2} = \begin{pmatrix} 1.1 & 0 \\ 0.2 & 0.8 \end{pmatrix}$$

$$C_{2,1} = \begin{pmatrix} 0 & 0 \end{pmatrix} \qquad C_{2,2} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

$$A_{3,1} = \begin{pmatrix} 0.11 & 0.09 \\ 0.09 & 0.07 \end{pmatrix} \qquad A_{3,2} = \begin{pmatrix} 0.32 & 0.30 \\ 0.09 & 0.38 \end{pmatrix}$$

$$C_{3,1} = \begin{pmatrix} 0 & 1 \end{pmatrix} \qquad C_{3,2} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

The second subsystem is linear, but it is unstable and unobservable. The second local model of the third subsystem is again unstable and unobservable. Due to this, methods available in the literature yield unfeasible LMIs. However, using Lemma 2.10 to formulate LMI conditions for Theorem 7.4, we obtain

$$P_{1,2,1} = \begin{pmatrix} 4.69 & -3.97 \\ -3.97 & 7.79 \end{pmatrix} \qquad P_{1,2,2} = \begin{pmatrix} 4.69 & -3.97 \\ -3.97 & 7.79 \end{pmatrix}$$

$$P_{2,3,1} = \begin{pmatrix} 3.45 & -3.35 \\ -3.35 & 5.05 \end{pmatrix} \qquad P_{2,3,2} = \begin{pmatrix} 4.09 & -3.91 \\ -3.91 & 6.41 \end{pmatrix}$$

$$P_{3,1,1} = \begin{pmatrix} 3.48 & -3.97 \\ -3.97 & 5.18 \end{pmatrix} \qquad P_{3,1,2} = \begin{pmatrix} 4.74 & -5.67 \\ -5.67 & 7.39 \end{pmatrix}$$

$$H_{1,1} = \begin{pmatrix} 4.83 & -2.45 \\ -2.50 & 6.26 \end{pmatrix} \qquad H_{1,2} = \begin{pmatrix} 3.77 & -3.21 \\ -1.84 & 6.46 \end{pmatrix}$$

$$H_{2,1} = \begin{pmatrix} 3.57 & -3.21 \\ -2.54 & 5.35 \end{pmatrix} \qquad H_{2,2} = \begin{pmatrix} 3.57 & -3.21 \\ -2.54 & 5.35 \end{pmatrix}$$

$$H_{3,1} = \begin{pmatrix} 4.55 & -5.05 \\ 0.45 & 5.84 \end{pmatrix} \qquad H_{3,2} = \begin{pmatrix} 4.14 & -4.61 \\ -4.85 & 6.27 \end{pmatrix}$$

$$L_{1,1} = \begin{pmatrix} -2.14 \\ 12.08 \end{pmatrix} \qquad L_{1,2} = \begin{pmatrix} 2.40 \\ 3.45 \end{pmatrix}$$

$$L_{3,1} = \begin{pmatrix} 2.42 \\ 4.27 \end{pmatrix} \qquad L_{3,2} = \begin{pmatrix} 3.89 \\ 7.93 \end{pmatrix}$$

Note that there are no observer gains $L_{2,1}$ and $L_{2,2}$. This is because the second subsystem on its own is not observable. However, this observer is able to estimate the states of the switching system above. A trajectory of the error dynamics for the case when $\boldsymbol{x}(0) = [-1, 1]^T$ and $\widehat{\boldsymbol{x}}(0) = 0$ is illustrated in Figure 7.4. For the simulation, the membership functions used were $h_1 = \frac{1}{2}(1 - \sin x_1)$, $h_2 = 1 - h_1$, and for $L_{2,1}$ and $L_{2,2}$ zero matrices were used. The initial subsystem was the first one, from which the system switched according to the periodic law. □

Let us now discuss the conditions developed for the α-sample variation of the Lyapunov function. In the example above, the conditions required that the Lyapunov function decreases with every switch/every sample, i.e., for all $(v_i, v_j) \in \mathscr{E}$. That is,

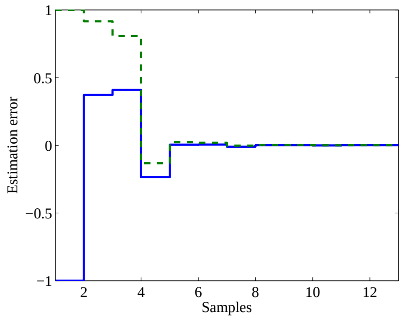


Figure 7.4: Estimation error for Example 7.2.

we had the conditions:

$$\begin{pmatrix} -P_{1,2,z} & (*) \\ H_{2,z}A_{2,z} - L_{2,z}C_{2,z} & -H_{2,z} + (*) + P_{2,3,z+1} \end{pmatrix} < 0$$
$$\begin{pmatrix} -P_{2,3,z} & (*) \\ H_{3,z}A_{3,z} - L_{3,z}C_{3,z} & -H_{3,z} + (*) + P_{3,1,z+1} \end{pmatrix} < 0$$
$$\begin{pmatrix} -P_{3,1,z} & (*) \\ H_{1,z}A_{1,z} - L_{1,z}C_{1,z} & -H_{1,z} + (*) + P_{1,2,z+1} \end{pmatrix} < 0$$

On the other hand, a 2-sample variation means that the Lyapunov function has to decrease along paths of length 2, i.e., we have the conditions:

$$\begin{pmatrix} -P_{1,2,z} & (*) & (*) \\ \begin{pmatrix} H_{2,z}A_{2,z} \\ -L_{2,z}C_{2,z} \end{pmatrix} & -H_{2,z} + (*) & (*) \\ 0 & \begin{pmatrix} H_{3,z+1}A_{3,z+1} \\ -L_{3,z+1}C_{3,z+1} \end{pmatrix} & \begin{pmatrix} -H_{3,z+1} + (*) \\ +P_{3,1,z+2} \end{pmatrix} \end{pmatrix} < 0$$
$$\begin{pmatrix} -P_{2,3,z} & (*) & (*) \\ \begin{pmatrix} H_{3,z}A_{3,z} \\ -L_{3,z}C_{3,z} \end{pmatrix} & -H_{3,z} + (*) & (*) \\ 0 & \begin{pmatrix} H_{1,z+1}A_{1,z+1} \\ -L_{1,z+1}C_{1,z+1} \end{pmatrix} & \begin{pmatrix} -H_{1,z+1} + (*) \\ +P_{1,2,z+2} \end{pmatrix} \end{pmatrix} < 0$$
$$\begin{pmatrix} -P_{3,1,z} & (*) & (*) \\ \begin{pmatrix} H_{1,z}A_{1,z} \\ -L_{1,z}C_{1,z} \end{pmatrix} & -H_{1,z} + (*) & (*) \\ 0 & \begin{pmatrix} H_{2,z+1}A_{2,z+1} \\ -L_{2,z+1}C_{2,z+1} \end{pmatrix} & \begin{pmatrix} -H_{2,z+1} + (*) \\ +P_{2,3,z+2} \end{pmatrix} \end{pmatrix} < 0$$

Furthermore, a 3-sample variation means that the Lyapunov function has to decrease along paths of length 3, which, in this case, is equivalent to the whole period of switching.

In the example above, one of the reasons for which the LMIs are feasible is that the subsystems may be active for a finite number of samples. In the case considered above, the switching occurs at every sample, i.e., each subsystem is active for one sample. Feasible LMIs are also obtained if the subsystems may be active for at most 2 samples. However, assuming that each subsystem, once activated may remain active for an infinite number of samples, the corresponding LMIs become unfeasible.

Let us now consider a more complex example.

Example 7.3 Consider a TS system with three subsystems, each having two local models, as follows:

$$A_{1,1} = \begin{pmatrix} 0.73 & 0.52 \\ 0.66 & 0.74 \end{pmatrix} \qquad A_{1,2} = \begin{pmatrix} 0.28 & 0.27 \\ 1.50 & 1.38 \end{pmatrix}$$
$$C_{1,1} = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad C_{1,2} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$
$$A_{2,1} = \begin{pmatrix} 1.05 & 0 \\ 0.5 & 0.1 \end{pmatrix} \qquad A_{2,2} = A_{2,1}$$
$$C_{2,1} = \begin{pmatrix} 0 & 0 \end{pmatrix} \qquad C_{2,2} = C_{2,1}$$
$$A_{3,1} = \begin{pmatrix} 0.86 & 0.72 \\ 0.23 & 0.76 \end{pmatrix} \qquad A_{3,2} = \begin{pmatrix} 0.25 & 1.51 \\ 1.94 & 1.83 \end{pmatrix}$$
$$C_{3,1} = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad C_{3,2} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

Subsystem 1 may be active for at most 5 samples, while subsystems 2 and 3 are active only for 1 sample, i.e., $p_1^m = 1$, $p_1^M = 5$, $p_2^m = p_2^M = p_3^m = p_3^M = 1$. The possible switches are presented in Figure 7.5. All the local models are unstable. Next to this, the second subsystem and the second local model of the third subsystem are unobservable.

Due to the unobservable and unstable local models, neither the conditions developed using a common quadratic Lyapunov function, nor those based on a common nonquadratic Lyapunov functions are feasible.

Using the conditions of Theorem 7.4, with a common quadratic Lyapunov function, we obtain the gains

$$H_{1,1} = \begin{pmatrix} 0.70 & -0.30 \\ -0.08 & 0.29 \end{pmatrix} \qquad H_{1,2} = \begin{pmatrix} 0.90 & -0.16 \\ -0.33 & 0.19 \end{pmatrix}$$
$$H_{2,1} = \begin{pmatrix} 0.58 & -0.35 \\ -0.10 & 0.51 \end{pmatrix} \qquad H_{2,2} = H_{2,1}$$
$$H_{3,1} = \begin{pmatrix} 0.58 & -0.39 \\ 0.05 & 0.37 \end{pmatrix} \qquad H_{3,2} = \begin{pmatrix} 0.61 & -0.37 \\ -0.37 & 0.24 \end{pmatrix}$$
$$L_{1,1} = \begin{pmatrix} 0.52 \\ 0.37 \end{pmatrix} \qquad L_{1,2} = \begin{pmatrix} 0.01 \\ 0.47 \end{pmatrix}$$
$$L_{3,1} = \begin{pmatrix} 0.28 \\ 0.37 \end{pmatrix} \qquad L_{3,2} = \begin{pmatrix} 0.25 \\ 0.85 \end{pmatrix}$$

Since the second subsystem is not observable, there are no gains $L_{2,1}$ and $L_{2,2}$. Trajectories of the states and the error are presented in Figures 7.6(a) and 7.6(b). The true initial states were $\boldsymbol{x}_0 = [-1, 1]^T$ and the estimated initial states were $\boldsymbol{x}_0 = 0$. The corresponding switching law is given in Figure 7.6(c). For testing, the membership functions are assumed to depend on an exogenous measured signal, with h_1 being presented in Figure 7.6(d).

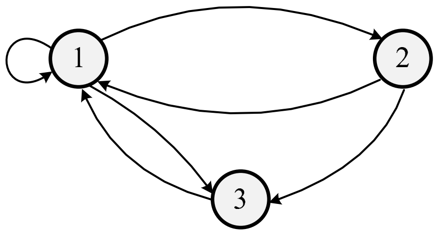


Figure 7.5: Graph representation of the switching system in Example 7.3.

Using the conditions developed with a nonquadratic Lyapunov function for the subsystems, but taking into account the possible switches (notably that subsystems 2 and 3 are active only for one sample), we obtain the gains

$$H_{1,1} = \begin{pmatrix} 0.66 & -0.30 \\ -0.09 & 0.31 \end{pmatrix} \qquad H_{1,2} = \begin{pmatrix} 0.95 & -0.16 \\ -0.36 & 0.20 \end{pmatrix}$$

$$H_{2,1} = \begin{pmatrix} 0.58 & -0.36 \\ -0.01 & 0.36 \end{pmatrix} \qquad H_{2,2} = H_{2,1}$$

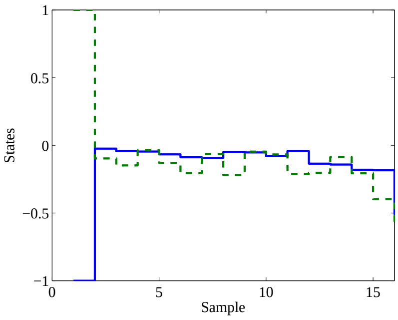
$$H_{3,1} = \begin{pmatrix} 0.59 & -0.41 \\ 0.06 & 0.35 \end{pmatrix} \qquad H_{3,2} = \begin{pmatrix} 0.65 & -0.39 \\ -0.38 & 0.25 \end{pmatrix}$$

$$L_{1,1} = \begin{pmatrix} 0.47 \\ 0.42 \end{pmatrix} \qquad L_{1,2} = \begin{pmatrix} 0.08 \\ 0.5 \end{pmatrix}$$

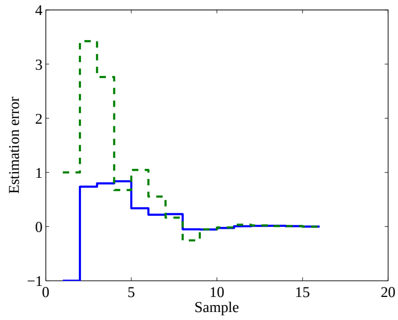
$$L_{3,1} = \begin{pmatrix} 0.28 \\ 0.37 \end{pmatrix} \qquad L_{3,2} = \begin{pmatrix} 0.23 \\ 0.83 \end{pmatrix}$$

which are quite close to those obtained by Theorem 7.4. It should be noted that without taking into account that subsystems 2 and 3 are active only for one sample, the conditions using a nonquadratic Lyapunov function for each subsystem are unfeasible. □

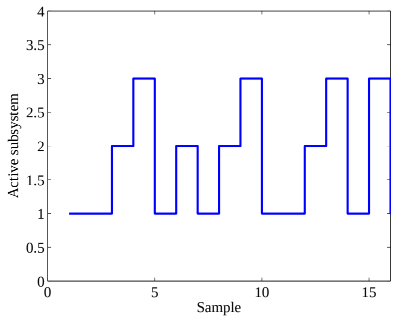
We assumed that the switching sequence is not known in advance and it cannot be directly influenced. Due to this assumption the conditions require that the estimation error dynamics is stable if the Lyapunov function decreases along every path of length α. This is the worst-case assumption, i.e., all possible combinations on



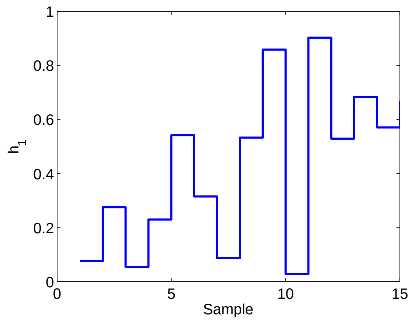
(a) A trajectory of the switching system.



(b) The trajectory of the error.



(c) The switching law used.



(d) Membership function.

Figure 7.6: Simulation results for Example 7.3.

switches between the estimation error subsystems have to be taken into account. If the switching sequence can be chosen, or it is known in advance, the conditions can be relaxed.

7.3 Observer design with controlled switches

Finally, let us consider the case when an observer has to be designed while the switches may be chosen. Therefore, our goal is to "stabilize" the error dynamics by suitably designed observer gains and by switching among the subsystems. The switching observer considered,

$$\begin{aligned} \widehat{\boldsymbol{x}}(k+1) &= A_{j,z}\widehat{\boldsymbol{x}}(k) + B_{j,z}\boldsymbol{u}(k) + H_{j,z}^{-1}L_{j,z}(\boldsymbol{y}(k) - \widehat{\boldsymbol{y}}(k)) \\ \widehat{\boldsymbol{y}}(k) &= C_{j,z}\widehat{\boldsymbol{x}}(k) \end{aligned} \tag{7.16}$$

is the same as before, but next to the matrices $H_{j,i}^{-1}$, $L_{j,i}$, $j = 1, 2 \ldots, n_s$, $i = 1, 2, \ldots, r$, the switching sequence also has to be determined.

The error dynamics is again

$$\boldsymbol{e}(k+1) = (A_{j,z} - H_{j,z}^{-1} L_{j,z} C_{j,z}) \boldsymbol{e}(k) \tag{7.17}$$

Since the switching sequence can be chosen, if there exists a subsystem with an asymptotically stable error dynamics that can be active for an infinite number of samples, the problem can be reformulated as finding a path – a classical graph theoretical problem – from each subsystem to this stable subsystem. If such a path exists then the switching law is given by this path, and the problem is solved. Therefore, we consider the case when none of the subsystems may be infinitely active. Next to this, for the easier development of the conditions, we assume that the associated graph is strongly connected, i.e., there exists a path between any two vertices.

Recall that we consider the associated directed graph $\mathscr{G} = \{\mathscr{V}, \mathscr{E}\}$, where the vertices $\mathscr{V} = \{v_1, v_2, \ldots, v_{n_s}\}$ correspond to the subsystems, and each edge $e_{i,j} = (v_i, v_j) \in \mathscr{E}$ corresponds to an admissible transition.

Similarly to the controller design case, we build a weight-adjacency matrix, that assigns to each admissible transition, including self-transitions, a weight. By convention, if $(v_i, v_j) \notin \mathscr{E}$, for $i \neq j$, $i, j = 1, 2, \ldots, n_s$ the corresponding weight $w_{i,j} = \infty$, and if $(v_i, v_i) \notin \mathscr{E}$, $i = 1, 2, \ldots, n_s$, then the corresponding weight $w_{i,i} = 1$. For all other edges, the corresponding weight will be given by an upper bound on the increase of a Lyapunov function associated to the error dynamics.

Consider the Lyapunov function

$$V = \boldsymbol{e}(k)^T P_{i,z} \boldsymbol{e}(k)$$

with $P_{i,z} = P_{i,z}^T > 0$, for the ith subsystem, $i = 1, 2, \ldots, n_s$.

Before constructing the weighting matrix, let us find constants $\delta_{i,j} > 0$ so that $V(\boldsymbol{e}(k+1)) \leq \delta_{i,j} V(\boldsymbol{e}(k))$, if the transition is (v_i, v_j), i.e., upper bounds on the increase of the Lyapunov function in one sample.

For any $(v_i, v_j) \in \mathscr{E}$ we have

$$\begin{aligned}
&V(\boldsymbol{e}(k+1)) - \delta_{i,j} V(\boldsymbol{e}(k)) < 0 \\
&\boldsymbol{e}(k+1)^T P_{j,z+1} \boldsymbol{e}(k+1) - \delta_{i,j} \boldsymbol{e}(k)^T P_{i,z} \boldsymbol{e}(k) < 0 \\
&\begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k+1) \end{pmatrix}^T \begin{pmatrix} -\delta_{i,j} P_{i,z} & 0 \\ 0 & P_{j,z+1} \end{pmatrix} \begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k+1) \end{pmatrix} < 0
\end{aligned}$$

At the same time, the system's dynamics can be written as

$$\begin{pmatrix} A_{i,z} - H_{i,z}^{-1} L_{i,z} C_{i,z} & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k+1) \end{pmatrix} = 0$$

Using Lemma 2.13, we have $V(\boldsymbol{e}(k+1)) - \delta_{i,j} V(\boldsymbol{e}(k)) < 0$ if there exists M so that

$$\begin{pmatrix} -\delta_{i,j} P_{i,z} & 0 \\ 0 & P_{j,z+1} \end{pmatrix} + M \begin{pmatrix} A_{i,z} - H_{i,z}^{-1} L_{i,z} C_{i,z} & -I \end{pmatrix} + (*) < 0$$

Choosing $M = \begin{pmatrix} 0 \\ H_{i,j,z} \end{pmatrix}$ we obtain the sufficient conditions

$$\begin{pmatrix} -\delta_{i,j} P_{i,z} & (*) \\ H_{i,j,z} A_{i,z} - L_{i,z} C_{i,z} & -H_{i,j,z} - H_{i,j,z}^T + P_{j,z+1} \end{pmatrix} < 0 \tag{7.18}$$

To find all $\delta_{i,j}$, one has to solve (7.18) for all $(v_i, v_j) \in \mathscr{E}$.

Now, define the weight matrix as $\mathscr{W} = [w_{i,j}]$ with

$$w_{i,j} = \begin{cases} \delta_{i,i}^{p_i^m - 1} & \text{if } \delta_{i,i} > 1 \\ \delta_{i,i}^{p_i^M - 1} & \text{if } \delta_{i,i} < 1 \\ \delta_{i,j} & \text{if } i \neq j \end{cases}$$

Further on, we follow the same reasoning as in Section 6.3. Assume that in the weight matrix constructed above there exists a subunitary cycle, i.e., there exists $\mathscr{C}_n = \{v_{c1}, \ v_{c2}, \ \ldots, v_{cp}, v_{c1}\}$ such that the product of the edges and nodes in this cycle is subunitary, and let this product be denoted by δ_n. For any subsystem i such that $v_i \in \mathscr{C}_n$, we have $V(\boldsymbol{e}_{k+n_c}) < \delta_n V(\boldsymbol{e}(k))$, i.e., after a full cycle, the corresponding Lyapunov function increases at most δ_n times, with $\delta_n < 1$. Consequently, the periodic switching law $\mathscr{C}_n = [c_1, c_2, \ldots, c_p, c_1]$ stabilizes the periodic error dynamics given by this cycle.

Let us now see the subsystems (if any) that are not in $\mathscr{C}_n$. Since we assumed that $\mathscr{G}$ is strongly connected, for all $v_i \notin \mathscr{C}$, there exists a path $\mathscr{P}(v_i, v_j)$ from the ith subsystem to a subsystem j on the cycle. Consider the switching law $\mathscr{P}(v_i, v_j)\mathscr{C}_n$, i.e., first a switching law that leads to the cycle and then the periodic switching law corresponding to the cycle. Although during the switches corresponding to $\mathscr{P}(v_i, v_j)$ the Lyapunov function might increase, during the periodic switching it will eventually decrease and converge to zero.

Based on the explanation above, the following result can be stated:

Theorem 7.8 *The error dynamics* (7.17) *are asymptotically stable along a switching law, if its associated graph* $\mathscr{G} = \{\mathscr{V}, \mathscr{E}, \mathscr{W}\}$ *contains a subunitary cycle* $\mathscr{C}_n$. *Furthermore, for the ith initial subsystem,* $i = 1, 2, \ldots, n_s$, *the switching law that stabilizes the error dynamics is given by* $\mathscr{P}(v_i, v_j)\mathscr{C}_n$, *where* v_i *denotes the vertex corresponding to the initial subsystems, and* $\mathscr{P}(v_i, v_j)$ *is a path to vertex* v_j, *with* $v_j \in \mathscr{C}_n$.

Similarly to Section 6.3, if the graph is not strongly connected, the possibility of developing an observer depends on the starting subsystem. Therefore, in order to design an observer, each strongly connected subgraph has to be analyzed.

7.4 Conclusions

In this chapter we considered observer design for switching nonlinear systems represented by TS fuzzy models. We have presented conditions that, when satisfied, guarantee that the estimation error converges to zero. The conditions were derived by taking into account the number of samples a subsystem may be active and the possible switches in the system.

We have considered three cases. The first case is when the switching is periodic. Second, we considered general switching systems, where the switching sequence is not known in advance and it cannot be directly influenced. Due to this assumption the conditions require that the estimation error dynamics decreases along the trajectory of the subsystems and the switches. This is the worst-case assumption, i.e., all possible combinations on switches between the estimation error subsystems have to be taken into account. Finally, we considered the case that next to designing the observer gains, the switching sequence can be chosen.

The developed conditions have been formulated as LMIs and extended to the α-sample variation of the Lyapunov function, in order to reduce their conservativeness. Other possibilities to relax the conditions represents the usage of double sums in the Lyapunov function, or using delayed Lyapunov functions or controllers and observers.

A shortcoming of the conditions is the computational complexity required to generate the possible combinations of switches and subsystems. In particular for large-scale switching systems, when considering α-sample variation, the number of conditions may be exponential in the number of subsystems, and in consequence, a large number of LMIs that has to be solved. However, reducing the conservativeness of the conditions by introducing additional sums in the Lyapunov function also increases the number of LMIs.

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Part III

Local analysis and synthesis

Chapter 8

Introduction and outline

8.1 Introduction

In this part, we consider the problem of analyzing local stability and developing local controller and observer design methods for discrete-time TS models while estimating a domain of attraction of the equilibrium point.

In the continuous-time case, the main inconvenient in the fuzzy Lyapunov function approach is that differentiating the Lyapunov function means differentiating the membership functions. Deriving tractable conditions while linking these derivatives to the system states is challenging. In (Tanaka et al., 2003), the proposed solution is to assume known upper bounds on the time derivative of the membership functions. First, this assumption induces local results: they are valid on the validity domain of the bounding assumption and a key point is then to enlarge this domain (Lee and Joo, 2014). Second, these upper bounds are far from being trivial to compute.

In the discrete time case, solutions are generally global. Since the time derivative of the Lyapunov candidate is no longer needed, non-quadratic Lyapunov functions have been introduced (Guerra and Vermeiren, 2004; Ding et al., 2006; Dong and Yang, 2009; Lee et al., 2011) for developing global stability and design conditions. More recently, by using Polya's theorem (Montagner et al., 2007; Sala and Ariño, 2007) asymptotically necessary and sufficient (ANS) LMI conditions have been obtained for stability in the sense of a chosen quadratic or nonquadratic Lyapunov function. Ding (2010) gave ANS stability conditions for both membership function-dependent model and membership function-dependent Lyapunov matrix. By increasing the complexity of the homogeneously polynomially parameter-dependent Lyapunov functions, in theory any sufficiently smooth Lyapunov function can be approximated. Unfortunately, the number of LMIs that have to be solved increase quickly, leading to numerical intractability (Zou and Yu, 2014). For all these results, if the developed conditions are feasible, then stability of the corresponding system is ensured globally – actually in the largest Lyapunov level set included in the domain where the TS model is defined. However, it is possible that stability (of the closed-loop system or error dynamics, as might be the case) cannot be ensured on the full domain where the TS model is defined, but by reducing this domain, the conditions become feasible (Pitarch et al., 2011).

For many nonlinear systems, only local stability of an equilibrium point can be established. This part addresses local stability analysis and design for discrete time TS models. In the literature few results exist on this specific topic. Some works deal with linear discrete systems with actuator and/or state saturations (da Silva and Tarbouriech, 2001, 2006), where the stability analysis is performed using quadratic Lyapunov functions while trying to find the maximum admissible quadratic domain of attraction. Others, considering the same problem of actuator saturation still using quadratic Lyapunov functions, proposed ways to design the domain of attraction based on convex set as polyhedrons (Hu et al., 2001) or as being saturation dependent (Cao and Lin, 2003), leading to slightly non-quadratic sets.

A quadratic domain of attraction can be restrictive compared to the true domain of attraction. Lee et al. (2013), developed a stability/stabilization approach to improve the construction of the domain of attraction. They developed a strategy for the stabilization of TS models using non-quadratic Lyapunov functions and an iterative procedure to maximize the domain of attraction. In (Lee and Joo, 2014), the main improvement remains in the use of homogeneous polynomial parameter-dependent matrices in the Lyapunov function as well as in the controller, which allow obtaining better results with a larger domain of attraction compared to (Lee et al., 2013). The first approach in (Lee et al., 2013) requires the knowledge of some a priori bounds on the variation of the membership functions over time, while in (Lee and Joo, 2014) the bounds on the derivative of the membership function with respect to the scheduling variables are introduced in the stability conditions. This procedure is complex and can be computationally expensive, especially for practical implementation.

The idea proposed in our work is to combine the advantageous tools existing in the discrete TS framework with the determination of a non-quadratic domain of attraction using an easy procedure requiring only the knowledge of the membership functions. The main interest is to ensure local asymptotic stability especially when no global one exists and allowing some good performances by proposing a way to construct and maximize the domain of attraction in a simple manner.

The material in this part is based on the following publications:

(P12) Zs. Lendek, Z. Nagy, J. Lauber, Local stabilization of discrete-time TS descriptor systems. Engineering Applications of Artificial Intelligence, vol. 67, pages 409-418, 2018.

(P13) B. Marx, Zs. Lendek, Local observer design for discrete-time TS systems. In Preprints of the 2017 IFAC World Congress, pages 873-878, Toulouse, France, July 2017.

(P14) Zs. Lendek, J. Lauber, Local stability of discrete-time TS fuzzy systems. In Proceedings of the 4th IFAC International Conference on Intelligent Control and Automation Sciences , pages 7-12, Reims, France, June 2016.

(P15) Zs. Lendek, J. Lauber, Local quadratic and nonquadratic stabilization of discrete-time TS fuzzy systems. In Proceedings of the 2016 IEEE World Congress on Computational Intelligence, pages 2182-2187, Vancouver, Canada, July 2016.

8.2 A motivating example

To motivate the research presented in this part, consider the following example.

Example 8.1 Consider the nonlinear system:

$$\begin{aligned} x_1(k+1) &= x_1^2(k) \\ x_2(k+1) &= x_1(k) + 0.5x_2(k) \end{aligned} \tag{8.1}$$

with $x_1(k) \in [-a, a]$, $a > 0$ being a parameter. It can be easily seen that (8.1) is locally stable for $x_1 \in (-1, 1)$.

The nonlinearity is x_1^2 and using the sector nonlinearity approach (Ohtake et al., 2001) on the domain $x_1(k) \in [-a, a]$, the resulting TS model is

$$\boldsymbol{x}(k+1) = h_1(x_1(k))A_1\boldsymbol{x} + h_2(x_1(k))A_2\boldsymbol{x}$$

with $h_1(x_1) = \frac{a-x_1(k)}{2a}$, $h_2(x_1(k)) = 1 - h_1(x_1(k))$, $A_1 = \begin{pmatrix} -a & 0 \\ 1 & 0.5 \end{pmatrix}$, $A_2 = \begin{pmatrix} a & 0 \\ 1 & 0.5 \end{pmatrix}$.

If $a < 1$, e.g., $a = 0.9$, the stability of the TS model can be easily proven e.g., using a common quadratic Lyapunov function.

If the sector nonlinearity approach is applied for $a > 1$, without including further conditions, no conclusion can be drawn regarding the stability of the TS model. A condition that leads to the feasibility of the associated LMI problem and thus makes it possible to draw some conclusion of local stability is e.g., $x_1^2(k+1) \leq 0.9x_1^2(k)$. However, the question on how to obtain such a condition and its exact interpretation remains open.

□

8.3 Notation

In this part we develop sufficient conditions for the local stability, local controller and local observer design of nonlinear discrete-time systems represented by Takagi-

Sugeno (TS) fuzzy models. For analysis we consider systems of the form

$$\begin{aligned} \boldsymbol{x}(k+1) &= \sum_{i=1}^{r} h_i(\boldsymbol{z}(k))A_i\boldsymbol{x}(k) \\ &= A_z\boldsymbol{x}(k) \end{aligned} \tag{8.2}$$

For local stabilization, the systems considered are of the form

$$\begin{aligned} \boldsymbol{x}(k+1) &= \sum_{i=1}^{r} h_i(\boldsymbol{z}_i(k))(A_i\boldsymbol{x}(k)+B_i\boldsymbol{u}(k)) \\ &= A_z\boldsymbol{x}(k)+B_z\boldsymbol{u}(k) \end{aligned} \tag{8.3}$$

and for observer design

$$\begin{aligned} \boldsymbol{x}(k+1) &= A_z\boldsymbol{x}(k)+B_z\boldsymbol{u}(k) \\ \boldsymbol{y}(k) &= C_z\boldsymbol{x}(k) \end{aligned} \tag{8.4}$$

In the above equations, $\boldsymbol{x}$ denotes the state vector, $\boldsymbol{u}$ is the input vector, r is the number of rules, $\boldsymbol{z}$ is the scheduling vector, h_i, $i = 1, 2, \ldots, r$ are normalized membership functions, and A_i, B_i, C_i, $i = 1, 2, \ldots, r$ are the local models.

All the above systems are assumed to be defined on a domain $\mathscr{D}$ including the origin. Although they may have several equilibrium points, in this part, without loss of generality, we consider the analysis of the origin, i.e., to develop conditions for the system, closed-loop system, and error dynamics, respectively to have a locally asymptotically stable equilibrium point in $\boldsymbol{x} = 0$ or $\boldsymbol{e} = 0$, respectively and to determine a region of attraction.

For this, we will make use of the following assumption

Assumption 8.1 *There exists a domain $\mathscr{D}_R \subset \mathscr{D}$ and a symmetric matrix function $R = R^T$ so that*

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T R \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} \geq 0$$

holds $\forall \boldsymbol{x}(k) \in \mathscr{D}_R$ or

$$\begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k+1) \end{pmatrix}^T R \begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k+1) \end{pmatrix} \geq 0$$

holds $\forall \boldsymbol{e}(k) \in \mathscr{D}_R$.

Note that this assumption can always be satisfied, e.g., by reducing $\mathscr{D}_R$ to the origin.

8.4 Outline

This part is organized as follows. Chapter 9 presents results regarding stability analysis, Chapter 10 conditions for stabilization and Chapter 11 conditions for observer design. In each chapter, the stability and design conditions, respectively, and the associated domain of attraction are given. Simulation examples illustrate the efficiency of the proposed methodology, and discussions are also provided.

Chapter 9

Stability analysis

This chapter considers local stability analysis of system (8.2), repeated here for convenience:

$$\boldsymbol{x}(k+1) = A_z \boldsymbol{x}(k) \tag{9.1}$$

defined on the domain $\mathscr{D}$ including the origin.

This chapter presents conditions which, when satisfied, ensure that the system (9.1) has a locally stable equilibrium point in the origin and determine a domain of attraction. To develop the conditions, first a quadratic Lyapunov function is considered. Afterwards, the conditions are generalized by using a nonquadratic Lyapunov function. The proposed approach is discussed and illustrated on numerical examples.

9.1 Local quadratic stability

In this section, we use a quadratic Lyapunov function to develop local stability conditions.

9.1.1 Stability conditions

Let us first assume that the domain $\mathscr{D}_R$ and a constant matrix R that satisfy Assumption 8.1 are given. The following result is straightforward.

Theorem 9.1 *The discrete-time nonlinear model* (9.1) *is locally asymptotically stable if there exist matrices* $P = P^T > 0$, M_i, $i = 1, 2, \ldots, r$ *and scalar* $\tau > 0$ *so that*

$$\begin{pmatrix} -P & (*) \\ M_z A_z & P - M_z - M_z^T \end{pmatrix} + \tau R < 0 \tag{9.2}$$

Moreover, the region of attraction , i.e., the region from which all trajectories converge to zero, includes $\mathscr{D}_S$*, where* $\mathscr{D}_S$ *is the largest Lyapunov level set included in* $\mathscr{D}_R$.

Proof: Consider the Lyapunov function $V = \boldsymbol{x}^T(k)P\boldsymbol{x}(k)$. The difference in the Lyapunov function is

$$\begin{aligned}\Delta V &= \boldsymbol{x}^T(k+1)P\boldsymbol{x}(k+1) - \boldsymbol{x}^T(k)P\boldsymbol{x}(k)\\ &= \begin{pmatrix}\boldsymbol{x}(k)\\ \boldsymbol{x}(k+1)\end{pmatrix}^T \begin{pmatrix}-P & 0\\ 0 & P\end{pmatrix}\begin{pmatrix}\boldsymbol{x}(k)\\ \boldsymbol{x}(k+1)\end{pmatrix}\end{aligned}$$

In the domain $\mathscr{D}_R$ Assumption 8.1 holds, thus, using Proposition 3, we have $\Delta V < 0$ if there exists $\tau > 0$ so that

$$\begin{pmatrix}\boldsymbol{x}(k)\\ \boldsymbol{x}(k+1)\end{pmatrix}^T \begin{pmatrix}-P & 0\\ 0 & P\end{pmatrix}\begin{pmatrix}\boldsymbol{x}(k)\\ \boldsymbol{x}(k+1)\end{pmatrix} + \tau \begin{pmatrix}\boldsymbol{x}(k)\\ \boldsymbol{x}(k+1)\end{pmatrix}^T R \begin{pmatrix}\boldsymbol{x}(k)\\ \boldsymbol{x}(k+1)\end{pmatrix} \leq 0$$

Furthermore, the dynamics (9.1) can be written as

$$\begin{pmatrix}A_z & -I\end{pmatrix}\begin{pmatrix}\boldsymbol{x}(k)\\ \boldsymbol{x}(k+1)\end{pmatrix} = 0$$

Using Lemma 2.13, we have $\Delta V < 0$ if there exist M so that

$$M\begin{pmatrix}A_z & -I\end{pmatrix} + (*) + \begin{pmatrix}-P & 0\\ 0 & P\end{pmatrix} + \tau R < 0$$

Choosing $M = \begin{pmatrix}0\\ M_z\end{pmatrix}$ leads to (9.2) and concludes the proof. ■

Sufficient LMI conditions can easily be derived using Lemma 2.10 or Lemma 2.11, as follows.

Corollary 9.2 *The discrete-time nonlinear model* (9.1) *is locally asymptotically stable if there exist matrices $P = P^T > 0$, M_i, $i = 1, 2, \ldots, r$ and scalar $\tau > 0$ so that the conditions of Lemma 2.10 or Lemma 2.11 hold, with*

$$\Gamma_{ij} = \begin{pmatrix}-P & (*)\\ M_iA_j & P - M_i - M_i^T\end{pmatrix} + \tau R$$

Moreover, the region of attraction includes $\mathscr{D}_S$, where $\mathscr{D}_S$ is the largest Lyapunov level set included in $\mathscr{D}_R$.

Now, the question arises what happens if R is unknown and how to determine R in this case. Note that – considering (9.2) – if R is also a decision variable, the parameter τ does not have any effect, as P, M_i, $i = 1, 2, \ldots, r$, τ and R are all decision variables. Thus, it is possible to solve (9.2) in the variables P, τR, and M_i $i = 1, 2, \ldots, r$. In what follows, with a slight abuse of notation, τR will be denoted simply by R. Then the following result can be formulated.

Theorem 9.3 *The discrete-time nonlinear model* (9.1) *is locally asymptotically stable in the domain* $\mathscr{D}_S$ *if there exist matrices* $P = P^T > 0$, M_i, $i = 1, 2, \ldots, r$ *and* $R = R^T$ *so that*

$$\begin{pmatrix} -P & (*) \\ M_z A_z & P - M_z - M_z^T \end{pmatrix} + R < 0 \tag{9.3}$$

where $\mathscr{D}_S$ *is the largest Lyapunov level set included in* $\mathscr{D}_R \bigcap \mathscr{D}$.

Proof: Consider the Lyapunov function $V = \boldsymbol{x}^T(k) P \boldsymbol{x}(k)$. The difference is

$$\Delta V = \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -P & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}$$

Let us assume that there exists a matrix $R = R^T$ and a domain $\mathscr{D}_R$ such that

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T R \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} \geq 0$$

holds $\forall \boldsymbol{x}(k) \in \mathscr{D}_R$. Note that since $\boldsymbol{x} = 0$ is an equilibrium point of the system (9.1), $\mathscr{D}_R$ always exists and it includes $\boldsymbol{x} = 0$. Then, we have $\Delta V < 0$ if

$$\Delta V < -\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T R \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}$$

i.e.,

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -P & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} + \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T R \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} < 0$$

Writing the dynamics of (9.1) as

$$\begin{pmatrix} A_z & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} = 0$$

and using Lemma 2.13, we have $\Delta V < 0$ if there exist M so that

$$M \begin{pmatrix} A_z & -I \end{pmatrix} + (*) + \begin{pmatrix} -P & 0 \\ 0 & P \end{pmatrix} + R < 0$$

Choosing $M = \begin{pmatrix} 0 \\ M_z \end{pmatrix}$ leads to (9.3). Wrt. the domain of attraction, recall that (9.1) has been defined in the domain $\mathscr{D}$, and that

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T R \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} \geq 0$$

holds in the domain $\mathscr{D}_R$. Thus, convergence is established for every trajectory starting in $\mathscr{D}_S$, where $\mathscr{D}_S$ is the largest Lyapunov level set contained in $\mathscr{D}_R \bigcap \mathscr{D}$. ■

Regarding the structure of R, several possibilities can be chosen, such as, $R = \begin{pmatrix} R_1 & 0 \\ 0 & -I \end{pmatrix}$, which establishes a direct relation between $\boldsymbol{x}(k)$ and $\boldsymbol{x}(k+1)$; a full R, which will give a more complex relation between two consecutive samples and thus a larger region, etc. Furthermore, depending on the structure of the system, specific structures can be chosen for R.

9.1.2 Example and discussion

Example 9.1 Recall the system in Example 8.1, repeated here for convenience:

$$\boldsymbol{x}(k+1) = h_1(x_1(k))A_1\boldsymbol{x} + h_2(x_1(k))A_2\boldsymbol{x}$$

with $h_1(x_1) = \frac{a - x_1(k)}{2a}$, $h_2(x_1(k)) = 1 - h_1(x_1(k))$, $A_1 = \begin{pmatrix} -a & 0 \\ 1 & 0.5 \end{pmatrix}$, $A_2 = \begin{pmatrix} a & 0 \\ 1 & 0.5 \end{pmatrix}$.

Let us assume that the TS model is defined for $a = 2$. Note that in this case, using classical conditions, it is not possible to establish (local) stability of the model. The following options are tested:

O_1: full R. The results are presented in Figure 9.1(a). The resulting matrix R is

$$R = \begin{pmatrix} 1.1 & 0 & 0 & -2.31 \\ 0 & 2.7 & 0 & -1.15 \\ 0 & 0 & -2.81 & 0 \\ -2.31 & -1.15 & 0 & 0 \end{pmatrix}$$

As it can be seen, the resulting condition involve both x_1 and x_2, and the resulting set is $\mathscr{D}_S = 0$.

O_2: $R = \begin{pmatrix} R_1 & 0 \\ 0 & -I \end{pmatrix}$. The results are presented in Figure 9.1(b). The resulting matrix R is

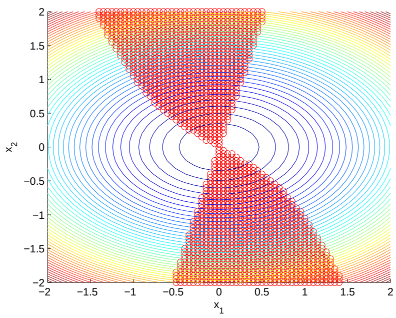
$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

i.e., it requires not only x_1 to decrease, but also x_2. This is why the resulting domain $\mathscr{D}_S = 0$.

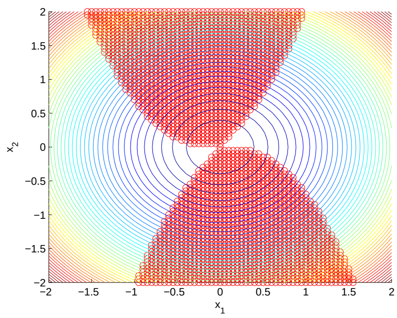
O_3: $R = \begin{pmatrix} R_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & R_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ giving the results in Figure 9.1(c). The resulting matrix R is

$$R = \begin{pmatrix} 0.18 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.18 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

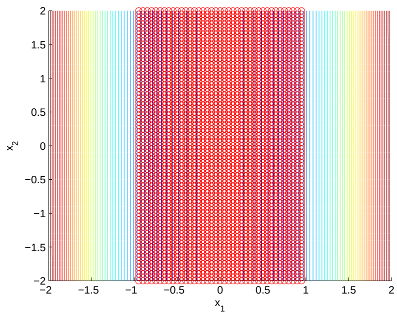
i.e., it requires only x_1 to decrease and actually gives the whole domain $\mathscr{D}_S = \{(x_1, x_2) | |x_1| < 1\}$.



(a) Results using a full R as in O_1.



(b) Results using the R in O_2.



(c) Results using the specific R in O_3.

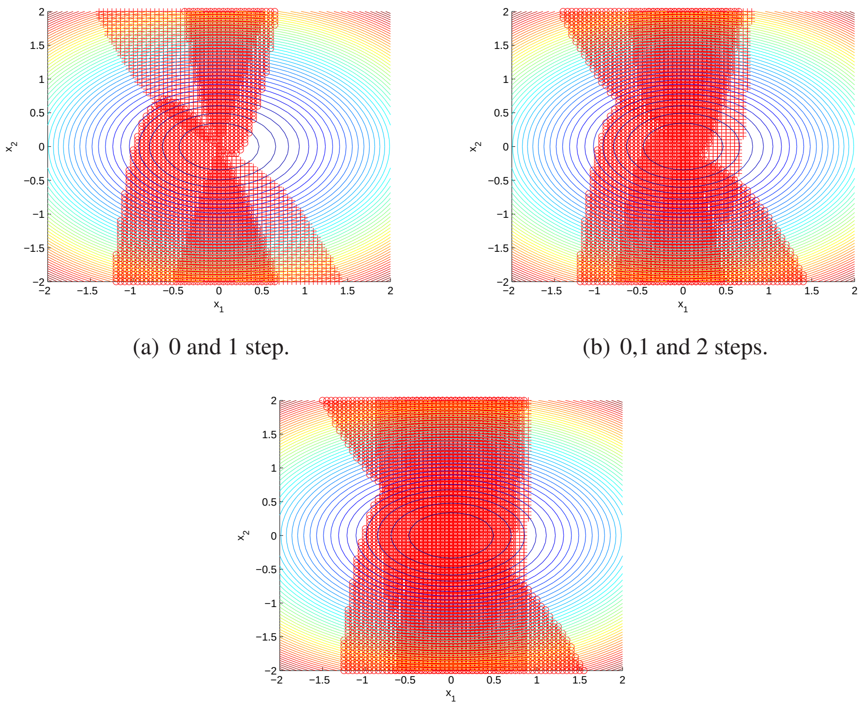
Figure 9.1: Results for Example 9.1.

For all the cases above, in order to obtain the maximum domain $\mathscr{D}_R$, the trace of R has been maximized. In Fig. 9.1, the ellipses denote the Lyapunov level sets, while 'o' stands for those points that satisfy Assumption 8.1 with the obtained R. □

Remark: In order for the conditions of Theorem 9.3, it is clear that R should have a structure to match the difference of the Lyapunov function, and ultimately the nonlinearities in the system model. On the other hand, this is a purely deterministic system, thus any trajectory that eventually gets in the domain $\mathscr{D}_S$ will converge.

Let us now look at those trajectories that do not start here, but arrive in $\mathscr{D}_R$.

For this, consider a full R. If we check the points that are in $\mathscr{D}_P$ plus those that get to $\mathscr{D}_R$ in one step, we have for Example 9.1 the results in Figure 9.2(a), for two steps the results in Figure 9.2(b) and for 3 steps the results in Figure 9.2(c).



(a) 0 and 1 step.

(b) 0,1 and 2 steps.

(c) 0,1, 2, and 3 steps.

Figure 9.2: Trajectories that converge in several steps to $\mathscr{D}_R \bigcap \mathscr{D}_P$ for Example 9.1.

As can be seen, in 3 steps we obtain almost the whole domain on x_1, although not on x_2. This is because the Lyapunov function is a common quadratic one and it does not include knowledge on the structure of the nonlinearities.

However, as shown above, looking to $\bigcup_{i=1}^{\beta} \mathscr{D}_{R_i}$, where $\mathscr{D}_{R_i}$ denotes the domain of those states whose trajectory will arrive in $\mathscr{D}_R$ in i steps and with β a relatively small, finite value can significantly improve the result.

Remark: Depending on the system considered, using $M = [M_{1z}^T, M_{2z}^T]^T$ instead of $M = [0, M_z^T]^T$ may improve the result. At this point, in order to reduce the number of variables involved and to be in line with results in the literature, we use $M = [0, M_z^T]^T$.

9.2 Local nonquadratic stability

Let us now consider a nonquadratic Lyapunov function $V = \boldsymbol{x}^T(k)P_z\boldsymbol{x}(k)$. Similarly to the case of common quadratic Lyapunov function, the following result can be established.

Theorem 9.4 *The equilibrium point* $\mathbf{x} = 0$ *of the discrete-time nonlinear model* (9.1) *is locally asymptotically stable if there exist matrices* $P_i = P_i^T > 0$, M_i, N_i, $i = 1, 2, \ldots, r$ *so that*

$$\begin{pmatrix} N_zA_z + (*) - P_z & (*) \\ M_zA_z - N_z & P_{z+1} - M_z - M_z^T \end{pmatrix} + R < 0 \tag{9.4}$$

Moreover, the region of attraction, i.e., the region from which all trajectories converge to zero, includes $\mathscr{D}_S$*, where* $\mathscr{D}_S$ *is the largest Lyapunov level set included in* $\mathscr{D}_R$.

Proof: Consider the Lyapunov function $V = \boldsymbol{x}^T(k)P_z\boldsymbol{x}(k)$. The difference is

$$\begin{aligned}\Delta V &= \boldsymbol{x}^T(k+1)P_{z+1}\boldsymbol{x}(k+1) - \boldsymbol{x}^T(k)P_z\boldsymbol{x}(k) \\ &= \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -P_z & 0 \\ 0 & P_{z+1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}\end{aligned}$$

In the domain $\mathscr{D}_R$ Assumption 8.1 holds, thus, using Proposition 3, we have $\Delta V < 0$ if

$$\Delta V < -\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T R \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}$$

i.e.,

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -P_z & 0 \\ 0 & P_{z+1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} + \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T R \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} < 0$$

Writing the dynamics of (9.1) as

$$\begin{pmatrix} A_z & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}^T(k+1) \end{pmatrix} = 0$$

and using Lemma 2.13, we have $\Delta V < 0$ if there exist M so that

$$M\begin{pmatrix} A_z & -I \end{pmatrix} + (*) + \begin{pmatrix} -P_z & 0 \\ 0 & P_{z+1} \end{pmatrix} + R < 0$$

Choosing $M = \begin{pmatrix} N_z \\ M_z \end{pmatrix}$ leads to (9.4). Wrt. the domain of attraction, recall that (9.1) has been defined in the domain $\mathscr{D}$, and that

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T R \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} \geq 0$$

holds in the domain $\mathscr{D}_R$. Thus, convergence is established for every trajectory starting in $\mathscr{D}_S$, where $\mathscr{D}_S$ is the largest Lyapunov level set contained in $\mathscr{D}_R \bigcap \mathscr{D}$. ■

Sufficient LMI conditions can easily be formulated as follows.

Corollary 9.5 *The discrete-time nonlinear model* (9.1) *is locally asymptotically stable if there exist matrices M_i, $i = 1, 2, \ldots, r$, $P = P^T > 0$, and $R = R^T$ so that Lemmas 2.10 or 2.11 hold, with*

$$\Gamma_{ij} = \begin{pmatrix} N_i A_j + (*) - P_i & (*) \\ M_i A_j - N_i & P_k - M_i - M_i^T \end{pmatrix} + R < 0$$

Moreover, the region of attraction includes $\mathscr{D}_S$, where $\mathscr{D}_S$ is the largest Lyapunov level set included in $\mathscr{D}_R$.

Similarly to the previous case, several possibilities can be chosen for R, such as diagonal, block-diagonal, a full one or any other structure. For instance, considering Example 9.1 and choosing $R = \begin{pmatrix} R_1 & 0 \\ 0 & -I \end{pmatrix}$, the resulting matrix R is

$$R = \begin{pmatrix} 2 & 0.5 & 0 & 0 \\ 0.5 & 0.25 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

This result is graphically illustrated in Figure 9.3. As can be seen, almost the entire domain is recovered.

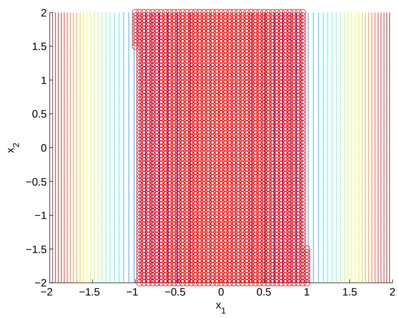


Figure 9.3: Domain recovered with the conditions of Theorem 9.4 for Example 9.1.

9.3 Generalization

In what follows, we generalize the previous results using the notation from Chapter 3.

With these notations, the system (9.1) is denoted as

$$\boldsymbol{x}(k+1) = \mathbb{A}_{G_0^A}\boldsymbol{x}(k) \tag{9.5}$$

with $G_0^A = \{0\}$.

9.3.1 Stability conditions

Let us now consider the nonquadratic Lyapunov function $V = \boldsymbol{x}^T(k)\mathbb{P}_{G_0^P}\boldsymbol{x}(k)$, where G_0^P contains the multiindex used in the Lyapunov matrix at time k. Furthermore, in order to also consider a nonquadratic domain, let us consider the constraint

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \mathbb{R}_{G_0^R} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} \geq 0$$

between consecutive samples. Then, following the lines of Theorem 9.3, we have the result:

Theorem 9.6 *The discrete-time nonlinear model* (9.5) *is locally asymptotically stable in the domain $\mathscr{D}_S$, if there exist matrices $P_{\boldsymbol{i}_j^P} = P_{\boldsymbol{i}_j^P}^T$, $\boldsymbol{i}_j^P = pr^{\boldsymbol{i}}_{G_j^P}$, $M_{\boldsymbol{i}_j^H}$, $\boldsymbol{i}_j^M = pr^{\boldsymbol{i}}_{G_j^M}$, and $N_{\boldsymbol{i}_j^H}$, $\boldsymbol{i}_j^N = pr^{\boldsymbol{i}}_{G_j^N}$, $\boldsymbol{i} \in \mathbb{I}_{G_V}$, $j = 0, 1$, and $R_{\boldsymbol{i}_0^R} = R_{\boldsymbol{i}_0^R}^T$, $\boldsymbol{i}_0^R = pr^{\boldsymbol{i}}_{G_0^R}$, where $G_V = G_0^P \cup G_1^P \cup (G_0^M \oplus G_0^A) \cup (G_0^N \oplus G_0^A) \cup G_0^R$ so that*

$$\begin{pmatrix} \mathbb{N}_{G_0^N}\mathbb{A}_{G_0^A} + (*) - \mathbb{P}_{G_0^P} & (*) \\ \mathbb{M}_{G_0^M}\mathbb{A}_{G_0^A} + \mathbb{N}_{G_0^N} & \mathbb{P}_{G_1^P} - \mathbb{M}_{G_0^M} - \mathbb{M}_{G_0^M}^T \end{pmatrix} + \mathbb{R}_{G_0^R} < 0 \tag{9.6}$$

where $\mathscr{D}_S$ is the largest Lyapunov level set included in $\mathscr{D}_R \bigcap \mathscr{D}$.

Sufficient LMI conditions can easily be derived for the above conditions. However, in order to relaxations such as Lemmas 2.10 or 2.11, to be efficiently applied and the computational complexity to be reduced, the delays used in the Lyapunov function and in the Finsler matrix should be suitably chosen. We will look at the terms that appear in G_V.

Let us first consider G_0^R. Since $\mathbb{R}_{G_0^R}$ is completely free and – when arriving to sufficient LMI conditions – each term in it will be added to any term in the sum in the expression

$$\begin{pmatrix} \mathbb{N}_{G_0^N}\mathbb{A}_{G_0^A} + (*) - \mathbb{P}_{G_0^P} & (*) \\ \mathbb{M}_{G_0^M}\mathbb{A}_{G_0^A} + \mathbb{N}_{G_0^N} & \mathbb{P}_{G_1^P} - \mathbb{M}_{G_0^M} - \mathbb{M}_{G_0^M}^T \end{pmatrix} \tag{9.7}$$

In order to reduce the number of final sums, G_0^R should contain all the indices appearing in (9.7). This means that in Theorem 9.6, $G_V = G_0^P \cup G_1^P \cup (G_0^M \oplus G_0^A) \cup$

$(G_0^N \oplus G_0^A) \cup G_0^R$ will become simply $G_V = G_0^P \cup G_1^P \cup (G_0^M \oplus G_0^A) \cup (G_0^N \oplus G_0^A)$. Second, since both $\mathbb{M}_{G_0^M}$ and $\mathbb{N}_{G_0^N}$ are multiplied by $\mathbb{A}_{G_0^A}$, to reduce the number of sums, the indices can be chosen the same, i.e., $G_0^M = G_0^N$. Thus, G_V becomes $G_V = G_0^P \cup G_1^P \cup (G_0^M \oplus G_0^A)$.

Further on, the reasoning from (Lendek et al., 2015) can be followed. Since for classic TS models $G_0^A = \{0\}$, to apply relaxations, $\mathbb{M}_{G_0^M}\mathbb{A}_{G_0^A}$ should contain a double sum in the same sample, for instance $\{0,0\}$. Choosing $G_0^P = \emptyset$ we recover the quadratic Lyapunov function $V = \boldsymbol{x}^T(k)P\boldsymbol{x}(k)$. Otherwise, since both G_0^P and G_1^P appear in (9.7), in order to combine at least one of them with $\mathbb{M}_{G_0^M}$ or $\mathbb{M}_{G_0^M}\mathbb{A}_{G_0^A}$ one can choose either $G_0^P = \{0\}$ (Guerra and Vermeiren, 2004) or $G_0^P = \{-1\}$ (Lendek et al., 2015). Then we have either

$$\begin{pmatrix} \mathbb{N}_0\mathbb{A}_0 + (*) - \mathbb{P}_0 & (*) \\ \mathbb{M}_0\mathbb{A}_0 + \mathbb{N}_0 & \mathbb{P}_1 - \mathbb{M}_0 - \mathbb{M}_0^T \end{pmatrix} + \mathbb{R}_{G_0^R} < 0 \tag{9.8}$$

or

$$\begin{pmatrix} \mathbb{N}_0\mathbb{A}_0 + (*) - \mathbb{P}_{-1} & (*) \\ \mathbb{M}_0\mathbb{A}_0 + \mathbb{N}_0 & \mathbb{P}_0 - \mathbb{M}_0 - \mathbb{M}_0^T \end{pmatrix} + \mathbb{R}_{G_0^R} < 0 \tag{9.9}$$

respectively. Both cases lead to three sums, a double one in 0 and a single one in either 1 or -1, thus a relaxation scheme can be applied on the double sum. Since G_0^R, as described above contains the same indices, the total number of sums will be three, with a double sum among them. Furthermore, since M is only used to prove stability, it is possible to add another sum to M that will not modify the total number of sums. Thus, we can add 1 in the case of (9.8) and -1 for (9.9), leading to

$$\begin{pmatrix} \mathbb{N}_{0,1}\mathbb{A}_0 + (*) - \mathbb{P}_0 & (*) \\ \mathbb{M}_{0,1}\mathbb{A}_0 + \mathbb{N}_{0,1} & \mathbb{P}_1 - \mathbb{M}_0 - \mathbb{M}_0^T \end{pmatrix} + \mathbb{R}_{G_0^R} < 0 \tag{9.10}$$

and

$$\begin{pmatrix} \mathbb{N}_{-1,0}\mathbb{A}_0 + (*) - \mathbb{P}_{-1} & (*) \\ \mathbb{M}_{-1,0}\mathbb{A}_0 + \mathbb{N}_{-1,0} & \mathbb{P}_0 - \mathbb{M}_0 - \mathbb{M}_0^T \end{pmatrix} + \mathbb{R}_{G_0^R} < 0 \tag{9.11}$$

respectively.

Now, any known relaxation (Sala and Ariño, 2007; Tuan et al., 2001) can be applied. More generally, the choice of the index sets has to favor in G_V multiple sums at the same samples. However, not every choice is adequate, for instance choosing $G_0^P = \emptyset$ and $G_0^M = \{-1,-1\}$ remains equivalent to quadratic stability.

In what follows we analyze how the negative delays in the Lyapunov function can be handled.

Recall that the domain $\mathscr{D}_S$ will be given by the largest Lyapunov level set in $\mathscr{D}$ and $\mathscr{D}_R$, i.e., the domain where Assumption 8.1 holds. In fact, see also (Pitarch et al., 2014), any trajectory that will arrive in $\mathscr{D}_S$ will eventually converge. Consequently, for a given $\boldsymbol{x}(0)$, if there exists $\beta > 1$ so that $\boldsymbol{x}(\beta) \in \mathscr{D}_S$, then $\boldsymbol{x}(k) \to 0$ as $k \to \infty$.

With this in mind, when using a Lyapunov function involving negative delays, instead of determining the level set of e.g., $V(\boldsymbol{x}(k)) = \boldsymbol{x}^T(k)\mathbb{P}_{-1}\boldsymbol{x}(k)$ – which would involve setting $\boldsymbol{x}(-1)$ – that is included on $\mathscr{D}_R$ it is possible to consider the level set of $\boldsymbol{x}^T(k+1)\mathbb{P}_0\boldsymbol{x}(k+1)$ that is included in $\mathscr{D}_{R_1}$.

Remark: The above is equivalent to using the Lyapunov function $V(\boldsymbol{x}(k)) = \boldsymbol{x}^T(k)\mathbb{P}_{G_0^P}\boldsymbol{x}(k)$ only for samples $k > \beta$, with $\beta \geq 0$ a finite number. This is possible because the TS model (9.5) is (locally) Lipschitz continuous, and thus it does not have a finite escape time.

Alternatively, consider the Lyapunov-like function

$$V(\boldsymbol{x}(k)) = \boldsymbol{x}(k+\beta)^T\mathbb{P}_{G_0^P}\boldsymbol{x}(k+\beta)$$

together with the domain $\mathscr{D}_R$ given by

$$\begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix}^T \mathbb{R}_{G_0^R} \begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix} > 0$$

$\forall \boldsymbol{x}(k+\beta) \in \mathscr{D}_R$, where G_0^P and G_0^R only contain 0 or positive delays. Note that this is not a Lyapunov function, as, depending on the system considered, $V(\boldsymbol{x}(k)) > 0$ $\forall \boldsymbol{x}(k) \neq 0$ cannot be ensured. However, $V(\boldsymbol{x}(k))$ is positive semidefinite.

The difference in the Lyapunov function is

$$\Delta V = \begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix}^T \begin{pmatrix} -\mathbb{P}_{G_0^P} & 0 \\ 0 & \mathbb{P}_{G_1^P} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix}$$

We have $\Delta V < 0$ if – considering the domain $\mathscr{D}_R$ –

$$\Delta V < -\begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix}^T \mathbb{R}_{G_0^R} \begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix}$$

i.e.,

$$\begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix}^T \begin{pmatrix} -\mathbb{P}_{G_0^P} & 0 \\ 0 & \mathbb{P}_{G_1^P} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix} + \begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix}^T \mathbb{R}_{G_0^R} \begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix} < 0$$

Writing the dynamics of (9.1) as

$$\begin{pmatrix} \mathbb{A}_{G_0^A} & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix} = 0$$

and using Lemma 2.13, we have $\Delta V < 0$ if there exist M so that

$$\begin{pmatrix} -\mathbb{P}_{G_0^P} & 0 \\ 0 & \mathbb{P}_{G_1^P} \end{pmatrix} + \mathbb{R}_{G_0^R} + M \begin{pmatrix} \mathbb{A}_{G_0^A} & -I \end{pmatrix} + (*) < 0$$

Choosing $M = \begin{pmatrix} 0 \\ \mathbb{M}_{G_0^M} \end{pmatrix}$ leads to

$$\begin{pmatrix} -\mathbb{P}_{G_0^P} & (*) \\ \mathbb{M}_{G_0^M}\mathbb{A}_{G_0^A} & \mathbb{P}_{G_1^P} - \mathbb{M}_{G_0^M} - \mathbb{M}_{G_0^M}^T \end{pmatrix} + \mathbb{R}_{G_0^R} < 0 \tag{9.12}$$

Note that from the point of view of the feasibility of the LMIs, under the same relaxation scheme, condition (9.12) is equivalent to (9.6) with $\mathbb{N}_{G_0^N} = 0$. However, both V and ΔV above are expressed in terms of $\boldsymbol{x}(k+\beta)$, not $\boldsymbol{x}(k)$. Nevertheless,

$$V(\boldsymbol{x}) = \boldsymbol{x}(k+\beta)^T \mathbb{P}_{G_0^P} \boldsymbol{x}(k+\beta)$$

is positive semidefinite and ΔV is negative semidefinite in $\boldsymbol{x}(k)$. Using Theorem 1 from (Grizzle and Kang, 2001), if condition (9.13) holds, then the trajectories will converge to the largest invariant set in $\Delta V = 0$. Since the largest invariant set is 0, local stability is established.

Regarding the domain of attraction, recall that (9.1) has been defined in the domain $\mathscr{D}$, and that

$$\begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix}^T \mathbb{R}_{G_0^R} \begin{pmatrix} \boldsymbol{x}(k+\beta) \\ \boldsymbol{x}(k+\beta+1) \end{pmatrix} > 0$$

holds in the domain $\mathscr{D}_R$. Thus, convergence is established for every trajectory that arrives after β samples in $\mathscr{D}_S$, where $\mathscr{D}_S$ is the largest Lyapunov level set contained in $\mathscr{D}_R \bigcap \mathscr{D}$.

All the above results can easily be extended using α-sample variation of the Lyapunov function (Kruszewski et al., 2008). In what follows, we illustrate this on the basic conditions from Theorem 9.3.

Recall that the system is given by

$$\boldsymbol{x}(k+1) = \mathbb{A}_{G_0^A} \boldsymbol{x}(k)$$

Consider – as in Section 9.3 – the nonquadratic Lyapunov function $V = \boldsymbol{x}^T(k)\mathbb{P}_{G_0^P}\boldsymbol{x}(k)$ and the domain $\mathscr{D}_R$ given by

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \\ \dots \\ \boldsymbol{x}(k+\alpha+1) \end{pmatrix}^T \mathbb{R}_{G_0^R} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \\ \dots \\ \boldsymbol{x}(k+\alpha+1) \end{pmatrix} > 0$$

on α consecutive samples. Then we can state the following result:

Theorem 9.7 *The discrete-time nonlinear model* (9.1) *is locally asymptotically stable in the domain* $\mathscr{D}_S$*, if there exist matrices* $P_{\boldsymbol{i}_j^P} = P_{\boldsymbol{i}_j^P}^T$*,* $\boldsymbol{i}_j^P = pr^{\boldsymbol{i}}_{G_j^P}$*, and* $M_{\boldsymbol{i}_j^H}$*,* $\boldsymbol{i}_j^M = pr^{\boldsymbol{i}}_{G_j^M}$*,* $\boldsymbol{i} \in \mathbb{I}_{G_V}$*,* $j = 0, 1, 2, \ldots, \alpha$*, and* $R_{\boldsymbol{i}_0^R} = R_{\boldsymbol{i}_0^R}^T$*,* $\boldsymbol{i}_0^R = pr^{\boldsymbol{i}}_{G_0^R}$*, where* $G_V = G_0^P \cup G_\alpha^P \cup \bigcup_{i=0}^{\alpha-1}(G_i^M \oplus G_i^A) \cup G_0^R$ *so that*

$$\begin{pmatrix} -\mathbb{P}_{G_0^P} & (*) & \ldots & (*) & (*) \\ \mathbb{M}_{G_0^M}\mathbb{A}_{G_0^A} & -\mathbb{M}_{G_0^M} - (*) & \ldots & (*) & (*) \\ 0 & \mathbb{M}_{G_1^M}\mathbb{A}_{G_1^A} & \ldots & (*) & (*) \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & \mathbb{M}_{G_{\alpha-1}^M}\mathbb{A}_{G_{\alpha-1}^A} & -\mathbb{M}_{G_{\alpha-1}^M} - (*) + \mathbb{P}_{G_\alpha^P} \end{pmatrix} + \mathbb{R}_{G_0^R} < 0 \tag{9.13}$$

where $\mathscr{D}_S$ *is the largest Lyapunov level set included in* $\mathscr{D}_R \bigcap \mathscr{D}$*.*

Remark: Note that G_V above denotes the multiset of all the delays that appear in the sum in (9.13). The terms in $\bigcup_{i=0}^{\alpha-1}(G_i^M \oplus G_i^A)$ are actually the delays that appear in the terms $\mathbb{M}_{G_i^M}\mathbb{A}_{G_i^A}$, $i = 1, 2, \ldots, \alpha - 1$.

9.3.2 Example and discussion

In this section we illustrate the stability conditions on a numerical example.

Example 9.2 Consider the nonlinear system:

$$\begin{aligned} x_1(k+1) &= 3sin(x_1(k))exp(x_1(k))x_1(k) \\ x_2(k+1) &= -x_1(k) + x_2^2(k) \end{aligned} \tag{9.14}$$

with $x_1(k), x_2(k) \in [-2, 2]$.

Using the sector nonlinearity approach (Ohtake et al., 2001), with both x_1 and x_2 as scheduling variables, the resulting TS model is

$$\boldsymbol{x}(k+1) = \sum_{i=1}^{4} h_i(\boldsymbol{z}(k))A_i\boldsymbol{x}(k)$$

with

$$A_1 = \begin{pmatrix} 3\sin(-\frac{\pi}{4})\exp(-\frac{\pi}{4}) & 0 \\ -1 & -2 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 3\sin(-\frac{\pi}{4})\exp(-\frac{\pi}{4}) & 0 \\ -1 & 2 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 3\sin(2)\exp(2) & 0 \\ -1 & -2 \end{pmatrix} \qquad A_4 = \begin{pmatrix} 3\sin(2)\exp(2) & 0 \\ -1 & 2 \end{pmatrix}$$

The origin is locally asymptotically stable, with the region of attraction given by $x_1 \in [-0.35, 0.25]$ and $x_2 \in (-1, 1)$. However, the (local) stability of the TS model cannot be proven using classical results. Furthermore, no domain can be recovered using a quadratic Lyapunov function.

In what follows, we consider a domain $\mathscr{D}_R$ with $\mathbb{R}_{G_0^R}$ having the form $\mathbb{R}_{G_0^R} = \begin{pmatrix} (*) & 0 \\ 0 & (*) \end{pmatrix}$, where the $(*)$ denotes values to be computed. This form – if not taking into account the indices in G_0^R – gives a direct relation between consecutive samples.

The following results have been obtained:

R_1: $G_0^P = \{0\}$, $G_0^M = \{0\}$, $G_0^R = \{0, 0, 1\}$: as can be seen in Figure 9.4(a), the domain is reduced to zero.

R_2: $G_0^P = \{-1\}$, $G_0^M = \{-1, 0\}$, $G_0^R = \{-1, 0, 0\}$: Figure 9.4(b) shows that some domain is recovered; this result also confirms that for some systems, a delayed Lyapunov function may provide less conservative results.

R_3: $G_0^P = \{0\}$, $G_0^M = \{0\}$, $G_0^R = \{-1, 0, 0, 1\}$: in order to increase the domain, we use a delayed index in R. As can be seen in Figure 9.4(c), the domain is considerably increased.

R_4: $G_0^P = \{-1, 0\}$, $G_0^M = \{-1, 0\}$, $G_0^R = \{-2, -1, 0, 0, 1\}$: the domain is again increased, as illustrated in Figure 9.4(d).

R_5: $G_0^P = \{-1, -1\}$, $G_0^M = \{-1, -1, 0\}$, $G_0^R = \{-3, -2, -1, -1, 0, 0\}$: the domain is again increased, as illustrated in Figure 9.4(e).

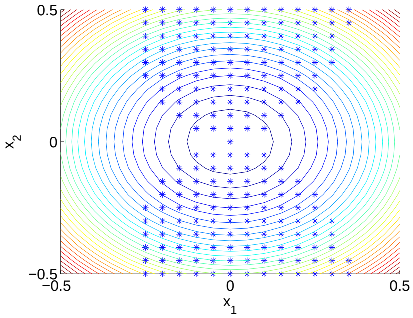
As illustrated in Figure 9.2, the choice of the indices G_0^R is more important than the choice of indices in the Lyapunov function. □

It should be noted that when using negative delays, the actual (corresponding to the current sample) domain cannot be graphically represented. Thus, in order to illustrate the domain $\mathscr{D}_R$ and the level sets, the "predicted" level sets and domain are represented. It can be seen that by increasing the number of delays used, the recovered region is increased (e.g., trajectories staring from the points recovered in Figure 9.4(e), even if not in the level set, all converge to zero and only these converge to zero). However, it has to be kept in mind that generally the implicit equation that defined $\mathscr{D}_R$ is very hard to solve.

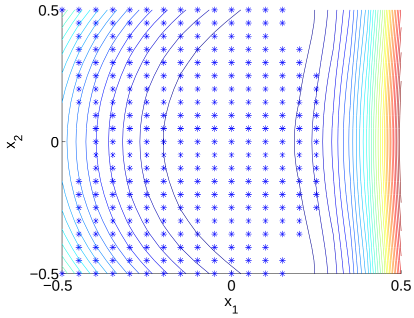
It should also be noted that in general using an unstructured/ full $\mathbb{R}_{G_0^R}$, although presenting extra degrees of freedom, may lead to a smaller region than e.g., a block-diagonal one.

9.4 Conclusions

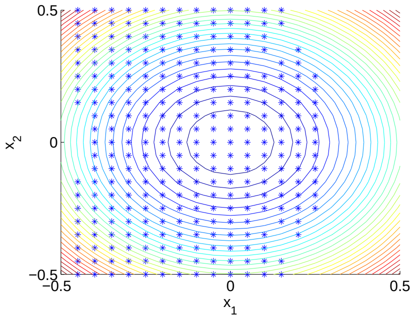
This chapter presented conditions to establish local stability of an equilibrium point of a TS fuzzy model. To develop the conditions, first a common quadratic Lyapunov



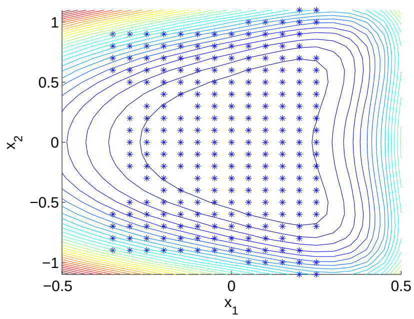
(a) Result for R_1



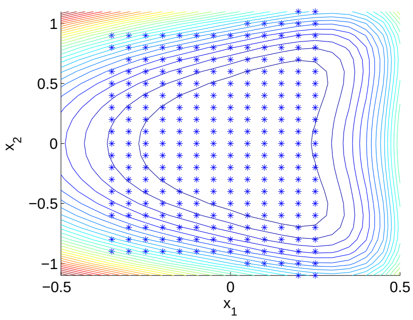
(b) Result for R_2



(c) Result for R_3



(d) Result for R_4



(e) Result for R_5

Figure 9.4: Results for Example 9.2.

function has been considered. Results have also been given using a nonquadratic Lyapunov function and generalized for delayed nonquadratic Lyapunov functions and α-sample variation of the Lyapunov functions. In all cases, an estimate of the region of attraction has also been obtained. The developed conditions were illustrated and discussed based on numerical example.

Chapter 10

Stabilization

This chapter considers local stabilization of system (8.3), repeated here for convenience:

$$\boldsymbol{x}(k+1) = A_z\boldsymbol{x}(k) + B_z\boldsymbol{u}(k) \tag{10.1}$$

defined on the domain $\mathscr{D}$ including the origin.

To develop conditions for the local stabilization, we first consider a PDC controller with a quadratic Lyapunov function and then generalize to nonquadratic Lyapunov functions and delayed controllers. Together with the stabilization a domain of attraction is also obtained. The presented approaches are discussed and illustrated on numerical examples.

10.1 Local quadratic stabilization

In this section, consider the controller design problem for the system (10.1). The controller used is

$$\boldsymbol{u}(k) = -F_z P^{-1}\boldsymbol{x}(k) \tag{10.2}$$

The closed-loop system can be expressed as

$$\boldsymbol{x}(k+1) = (A_z - B_z F_z P^{-1})\boldsymbol{x}(k) \tag{10.3}$$

Our goal is to develop conditions that ensure that this system has a locally asymptotically stable equilibrium point in $\boldsymbol{x} = 0$ and determine a region of attraction. For determining the stabilization conditions, first a quadratic Lyapunov function of the form $V = \boldsymbol{x}^T(k)P^{-1}\boldsymbol{x}(k)$ will be used.

10.1.1 Design conditions

Using a quadratic Lyapunov function and Assumption 8.1 for the closed-loop system, the following result can be stated.

Theorem 10.1 *The closed-loop system* (10.3) *is locally asymptotically stable if there exist matrices* $P = P^T > 0$, F_i, $i = 1, 2, \ldots, r$ *and* $W = W^T$ *so that*

$$\begin{pmatrix} -P & (*) \\ A_z P - B_z F_z & -P \end{pmatrix} + W < 0$$

holds. Moreover, the region of attraction includes $\mathscr{D}_S$*, where* $\mathscr{D}_S$ *is the largest Lyapunov level set included in* $\mathscr{D}_R$*, with* R *given by* $R = \begin{pmatrix} P^{-1} & 0 \\ 0 & P^{-1} \end{pmatrix} W \begin{pmatrix} P^{-1} & 0 \\ 0 & P^{-1} \end{pmatrix}$.

Proof: The difference in the Lyapunov function is

$$\Delta V = \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -P^{-1} & 0 \\ 0 & P^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}$$

and the closed-loop system can be expressed as

$$\begin{pmatrix} A_z - B_z F_z P^{-1} & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} = 0$$

Furthermore, in the domain $\mathscr{D}_R$, Assumption 8.1 holds. Thus, in this domain, (10.3) is locally asymptotically stable, if there exists M so that

$$M \begin{pmatrix} A_z - B_z F_z P^{-1} & -I \end{pmatrix} + (*) + \begin{pmatrix} -P^{-1} & 0 \\ 0 & P \end{pmatrix} + R < 0$$

Choosing

$$M = \begin{pmatrix} 0 \\ P^{-1} \end{pmatrix}$$

and congruence with

$$\begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}$$

leads to

$$\begin{pmatrix} -P & (*) \\ A_z P - B_z F_z & -P \end{pmatrix} + \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} R \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} < 0$$

Denoting $W = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} R \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}$ we obtain the conditions in Theorem 10.1. Since Assumption 8.1 holds only in the domain $\mathscr{D}_R$, the region of attraction includes the largest Lyapunov level set contained in $\mathscr{D}_R$. ■

Sufficient LMI conditions can easily be derived using Lemmas 2.11 or 2.10, as follows.

Corollary 10.2 *The closed-loop system* (10.3) *is locally asymptotically stable if there exist matrices $P = P^T > 0$, F_i, $i = 1, 2, \dots, r$ and $W = W^T$, so that Lemmas 2.10 or 2.11 hold, with*

$$\Gamma_{i,j} = \begin{pmatrix} -P & (*) \\ A_iP - B_iF_j & -P \end{pmatrix} + W < 0$$

Moreover, the region of attraction includes $\mathscr{D}_S$, where $\mathscr{D}_S$ is the largest Lyapunov level set included in $\mathscr{D}_R$, given by

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} P^{-1} & 0 \\ 0 & P^{-1} \end{pmatrix} W \times \begin{pmatrix} P^{-1} & 0 \\ 0 & P^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} > 0$$

10.1.2 Examples and discussion

Similarly to stability analysis, it should be noted that although Theorem 10.1 and Corollary 10.2 only establish convergence of the trajectories that start in $\mathscr{D}_S$, actually, any trajectory that eventually converges to $\mathscr{D}_S$ will converge to zero. Regarding the structure of W and, consequently, of R, several possibilities can be chosen, which will reflect different relations between consecutive states. To illustrate the use of the above conditions, and the effect that the structure of W will have on the result, consider the following example.

Example 10.1 Consider the nonlinear system

$$\begin{aligned} x_1(k+1) &= x_1^2(k) \\ x_2(k+1) &= x_1(k) + 0.5x_2(k) + u(k) \end{aligned} \tag{10.4}$$

with $x_1(k) \in [-2, 2]$, $a > 0$ being a parameter. It can be easily seen that (10.4) can only be stabilized if $x_1(k) \in (-1, 1)$.

Using the sector nonlinearity approach, the resulting TS model is

$$\boldsymbol{x}(k+1) = h_1(x_1(k))A_1\boldsymbol{x} + h_2(x_1(k))A_2\boldsymbol{x} + Bu(k)$$

with $h_1(x_1) = \frac{a - x_1(k)}{4}$, $h_2(x_1(k)) = 1 - h_1(x_1(k))$, $A_1 = \begin{pmatrix} -2 & 0 \\ 1 & 0.5 \end{pmatrix}$, $A_2 = \begin{pmatrix} 2 & 0 \\ 1 & 0.5 \end{pmatrix}$, and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Since the local models are not controllable, classical conditions for controller design fail. In what follows, our goal is to (locally) stabilize the system. For this, several structures of the matrix W are tested:

O_1: full W. The results are presented in Figure 10.1(a). The resulting matrix R is

$$R = 10^5 \begin{pmatrix} -1.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1.5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The resulting set $\mathscr{D}_S = 0$.

O_2: $W = \begin{pmatrix} W_1 & 0 \\ 0 & -I \end{pmatrix}$. The results are presented in Figure 10.1(b). The resulting matrix R is

$$R = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 0.006 & 0 & 0 \\ 0 & 0 & -41.79 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The domain has been significantly increased.

O_3: $W = \begin{pmatrix} W_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & W_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ giving the results in Figure 10.1(c). The resulting matrix R is

$$R = \begin{pmatrix} 0.0007 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.016 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In this case, the resulting domain is reduced wrt. to the previous one due to the shape of the Lyapunov level set.

For all the cases above, the trace of R has been maximized. As can be seen, the recovered region that is proven stable is actually much smaller than the region containing those initial points that actually converge to zero using the computed control law. In what follows, consider the block-diagonal W above and verify those points that converge to $\mathscr{D}_R$ in Figure 10.1(b) in one and two steps, respectively. The results are shown in Figures 10.2(a) and 10.2(b), respectively.

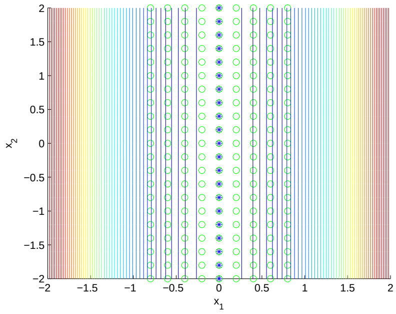
As can be seen, a significant improvement of the domain is obtained. In fact, considering $\bigcup_{i=1}^{\beta} \mathscr{D}_{R_i}$, where $\mathscr{D}_{R_i}$ denotes the domain of those states whose trajectory will arrive in $\mathscr{D}_R$ in i steps and with β a relatively small, finite value, can improve the result. □

A way to improve the results is by developing conditions based on non-quadratic Lyapunov function.

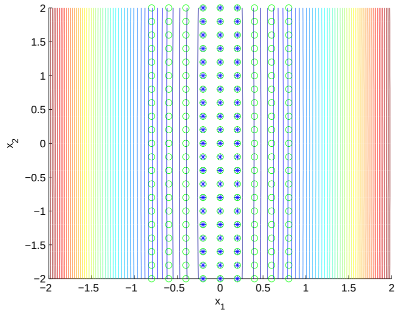
10.2 Local non-quadratic stabilization

In what follows, we consider a non-PDC controller of the form

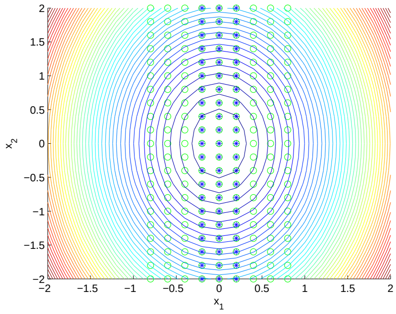
$$\boldsymbol{u}(k) = -F_z H_z^{-1} \boldsymbol{x}(k) \tag{10.5}$$



(a) Results using a full R.

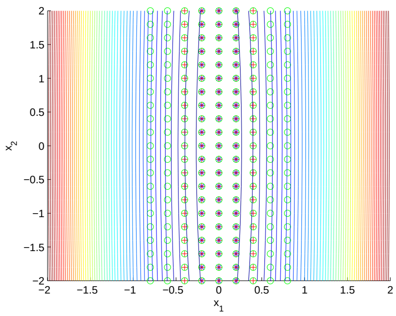


(b) Results using the R in O_2.

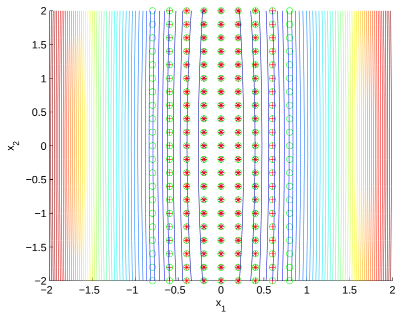


(c) Results using the specific R in O_3.

Figure 10.1: Results for Example 10.1: $\mathscr{D}_R$ (blue $*$), the Lyapunov level sets, and the points that will actually converge with the computed control (green o)



(a) 0 and 1 step.



(b) 0,1 and 2 steps.

Figure 10.2: Trajectories that converge to $\mathscr{D}_R$ for Example 10.1.

The closed-loop system can be expressed as

$$\boldsymbol{x}(k+1) = (A_z - B_z F_z H_z^{-1})\boldsymbol{x}(k) \tag{10.6}$$

Two Lyapunov functions will be considered, similarly to the results in (Lendek et al., 2015):

- Case 1: $V = \boldsymbol{x}^T(k)H_z^{-T}P_zH_z^{-1}\boldsymbol{x}(k)$
- Case 2: $V = \boldsymbol{x}^T(k)P_z\boldsymbol{x}(k)$

Let us first consider the Lyapunov function in the Case 1 above, together with a domain $\mathscr{D}_R$ defined as

$$\begin{pmatrix}\boldsymbol{x}(k)\\ \boldsymbol{x}(k+1)\end{pmatrix}^T \begin{pmatrix}H_z^{-T} & 0\\ 0 & H_{z+}^{-T}\end{pmatrix} R \begin{pmatrix}H_z^{-1} & 0\\ 0 & H_{z+}^{-1}\end{pmatrix}\begin{pmatrix}\boldsymbol{x}(k)\\ \boldsymbol{x}(k+1)\end{pmatrix} > 0 \tag{10.7}$$

Then, the following result can be established:

Theorem 10.3 *The closed-loop system* (10.6) *is locally asymptotically stable if there exist matrices* $P = P^T > 0$, F_i, H_i $i = 1, 2, \ldots, r$ *and* R *so that*

$$\begin{pmatrix}-P_z & (*)\\ A_zH_z - B_zF_z & -H_{z+} - H_{z+}^T + P_{z+}\end{pmatrix} + R < 0$$

holds. Moreover, the region of attraction includes $\mathscr{D}_S$*, where* $\mathscr{D}_S$ *is the largest Lyapunov level set included in* $\mathscr{D}_R$*.*

Proof: The difference in the Lyapunov function is

$$\Delta V = \begin{pmatrix}\boldsymbol{x}(k)\\ \boldsymbol{x}(k+1)\end{pmatrix}^T \begin{pmatrix}-H_z^TP_zH_z^T & 0\\ 0 & H_{z+}^TP_{z+}H_{z+}^T\end{pmatrix}\begin{pmatrix}\boldsymbol{x}(k)\\ \boldsymbol{x}(k+1)\end{pmatrix}$$

and the closed-loop system can be expressed as

$$\begin{pmatrix}A_z - B_zF_zH_z^{-1} & -I\end{pmatrix}\begin{pmatrix}\boldsymbol{x}(k)\\ \boldsymbol{x}(k+1)\end{pmatrix} = 0$$

Furthermore, in the domain $\mathscr{D}_R$, (10.7) holds. Thus, in this domain, (10.6) is locally asymptotically stable, if there exists M so that

$$M\begin{pmatrix}A_z - B_zF_zH_z^{-1} & -I\end{pmatrix} + (*) + \begin{pmatrix}-H_z^TP_zH_z^T & 0\\ 0 & H_{z+}^TP_{z+}H_{z+}^T\end{pmatrix}$$
$$+\begin{pmatrix}H_z^{-T} & 0\\ 0 & H_{z+}^{-T}\end{pmatrix} R \begin{pmatrix}H_z^{-1} & 0\\ 0 & H_{z+}^{-1}\end{pmatrix} < 0$$

Choosing $M = \begin{pmatrix}0\\ H_{z+}^{-T}\end{pmatrix}$ and congruence with $\begin{pmatrix}H_z^T & 0\\ 0 & H_{z+}^T\end{pmatrix}$ leads to the condition of Theorem 10.3. Furthermore, since the condition (10.7) holds only in the domain $\mathscr{D}_R$, the region of attraction includes the largest Lyapunov level set contained in $\mathscr{D}_R$. ■

Let us now consider the Lyapunov function in Case 2, together with a domain $\mathscr{D}_R$ defined as

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} H_z^{-T} & 0 \\ 0 & P_{z+}^{-1} \end{pmatrix} R \begin{pmatrix} H_z^{-1} & 0 \\ 0 & P_{z+}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} > 0 \tag{10.8}$$

For this case, the following result can be established:

Theorem 10.4 *The closed-loop system* (10.6) *is locally asymptotically stable if there exist matrices* $P = P^T > 0$, F_i, H_i $i = 1, 2, \ldots, r$ *and* R *so that*

$$\begin{pmatrix} -H_z - H_z^T + P_z & (*) \\ A_z H_z - B_z F_z & -P_{z+} \end{pmatrix} + R < 0$$

holds. Moreover, the region of attraction includes $\mathscr{D}_S$*, where* $\mathscr{D}_S$ *is the largest Lyapunov level set included in* $\mathscr{D}_R$*.*

Proof: The difference in the Lyapunov function is

$$\Delta V = \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -P_z^{-1} & 0 \\ 0 & P_{z+}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}$$

and the closed-loop system can be expressed as

$$\begin{pmatrix} A_z - B_z F_z H_z^{-1} & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} = 0$$

Furthermore, in the domain $\mathscr{D}_R$, (10.8) holds. Thus, in this domain, (10.6) is locally asymptotically stable, if there exists M so that

$$\begin{aligned} & M \begin{pmatrix} A_z - B_z F_z H_z^{-1} & -I \end{pmatrix} + (*) \\ & + \begin{pmatrix} -P_z^{-1} & 0 \\ 0 & P_{z+} \end{pmatrix} + \begin{pmatrix} H_z^{-T} & 0 \\ 0 & P_{z+}^{-1} \end{pmatrix} R \begin{pmatrix} H_z^{-1} & 0 \\ 0 & P_{z+}^{-1} \end{pmatrix} < 0 \end{aligned}$$

Choosing $M = \begin{pmatrix} 0 \\ P_{z+}^{-1} \end{pmatrix}$ and congruence with

$$\begin{pmatrix} H_z^T & 0 \\ 0 & P_{z+} \end{pmatrix}$$

leads to

$$\begin{pmatrix} -H_z^T P_z^{-1} H_z & (*) \\ A_z H_z - B_z F_z & -P_{z+} \end{pmatrix} + R < 0$$

Applying Property 5 gives the condition of Theorem 10.4. Furthermore, since the condition (10.7) holds only in the domain $\mathscr{D}_R$, the region of attraction includes the largest Lyapunov level set contained in $\mathscr{D}_R$. ■

Remark: Note that the conditions expressed in Theorems 10.3 and 10.4 above are not equivalent and do not include each other (see also (Lendek et al., 2015)). Depending on the system considered one or the other may give less conservative results.

Similarly to the quadratic case, several possibilities can be chosen for W, such as diagonal, block-diagonal, a full one or any other structure. For instance, considering Example 10.1, choosing a diagonal W, and using the conditions of Theorem 10.3, the resulting matrix W is $W = \text{diag}\{22.75, 29.67, -116.25, 3.47\}$ and the estimated domain of attraction is graphically illustrated in Figure 10.3. On the other hand, the result obtained using Theorem 10.4 are not better than those obtained with a quadratic Lyapunov function.

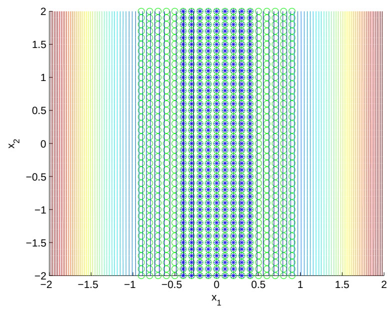


Figure 10.3: Domain recovered with the conditions of Theorem 10.3 for Example 10.1.

10.3 Generalization

In what follows, we will extend the results from the previous section using delayed nonquadratic Lyapunov functions and non-PDC controllers. Thus, consider the controller design problem for the system

$$\boldsymbol{x}(k+1) = \mathbb{A}_{G_0^A}\boldsymbol{x}(k) + \mathbb{B}_{G_0^B}\boldsymbol{u}(k) \tag{10.9}$$

using a controller of the form

$$\boldsymbol{u}(k) = -\mathbb{F}_{G_0^F}\mathbb{H}^{-1}_{G_0^H}\boldsymbol{x}(k) \tag{10.10}$$

The closed-loop system can be expressed as

$$\boldsymbol{x}(k+1) = (\mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}^{-1}_{G_0^H})\boldsymbol{x}(k) \tag{10.11}$$

10.3.1 Design conditions

The following two general Lyapunov functions will be considered:

- Case 1: $V = \boldsymbol{x}^T(k)\mathbb{H}_{G_0^H}^{-T}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1}\boldsymbol{x}(k)$ and
- Case 2: $V = \boldsymbol{x}^T(k)\mathbb{P}_{G_0^P}^{-1}\boldsymbol{x}(k)$ and

Let us first consider the Lyapunov function in the Case 1 above, together with a domain $\mathscr{D}_R$ defined as

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} \mathbb{H}_{G_0^H}^{-T} & 0 \\ 0 & \mathbb{H}_{G_1^H}^{-T} \end{pmatrix} \mathbb{R}_{G_0^R} \begin{pmatrix} \mathbb{H}_{G_0^H}^{-1} & 0 \\ 0 & \mathbb{H}_{G_1^H}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} > 0 \qquad (10.12)$$

Then, the following result can be established:

Theorem 10.5 *The closed-loop system* (10.11) *is locally asymptotically stable in the domain $\mathscr{D}_S$, if there exist matrices $P_{\boldsymbol{i}_j^P} = P_{\boldsymbol{i}_j^P}^T$, $\boldsymbol{i}_j^P = pr_{G_j^P}^{\boldsymbol{i}}$, $H_{\boldsymbol{i}_j^H}$, $\boldsymbol{i}_j^H = pr_{G_j^H}^{\boldsymbol{i}}$, and $F_{\boldsymbol{i}_j^F}$, $\boldsymbol{i}_j^F = pr_{G_j^F}^{\boldsymbol{i}}$, $\boldsymbol{i} \in \mathbb{I}_{G_V}$, $j = 0, 1$, and $R_{\boldsymbol{i}_0^R} = R_{\boldsymbol{i}_0^R}^T$, $\boldsymbol{i}_0^R = pr_{G_0^R}^{\boldsymbol{i}}$, where $G_V = G_0^P \cup G_1^P \cup (G_0^M \oplus G_0^A) \cup (G_0^F \oplus G_0^B) \cup G_0^R$ so that*

$$\begin{pmatrix} -\mathbb{P}_{G_0^P} & (*) \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & -\mathbb{H}_{G_1^H} - \mathbb{H}_{G_1^H}^T + \mathbb{P}_{G_1^P} \end{pmatrix} + \mathbb{R}_{G_0^R} < 0$$

holds. Moreover, the region of attraction includes $\mathscr{D}_S$, where $\mathscr{D}_S$ is the largest Lyapunov level set included in $\mathscr{D}_R$.

Proof: The difference in the Lyapunov function is

$$\Delta V = \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -\mathbb{H}_{G_0^H}^{-T}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1} & 0 \\ 0 & \mathbb{H}_{G_1^H}^{-T}\mathbb{P}_{G_1^P}\mathbb{H}_{G_1^H}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}$$

and the closed-loop system can be expressed as

$$\begin{pmatrix} \mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} = 0$$

Furthermore, in the domain $\mathscr{D}_R$, (10.12) holds. Thus, in this domain, (10.11) is locally asymptotically stable, if there exists M so that

$$\begin{pmatrix} -\mathbb{H}_{G_0^H}^{-T}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1} & 0 \\ 0 & \mathbb{H}_{G_1^H}^{-T}\mathbb{P}_{G_1^P}\mathbb{H}_{G_1^H}^{-1} \end{pmatrix} + M \begin{pmatrix} \mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} & -I \end{pmatrix} + (*)$$
$$+ \begin{pmatrix} \mathbb{H}_{G_0^H}^{-T} & 0 \\ 0 & \mathbb{H}_{G_1^H}^{-T} \end{pmatrix} \mathbb{R}_{G_0^R} \begin{pmatrix} \mathbb{H}_{G_0^H}^{-1} & 0 \\ 0 & \mathbb{H}_{G_1^H}^{-1} \end{pmatrix} < 0$$

Choosing

$$M = \begin{pmatrix} 0 \\ \mathbb{H}_{G_1^H}^{-T} \end{pmatrix}$$

and congruence with

$$\begin{pmatrix} \mathbb{H}_{G_0^H}^T & 0 \\ 0 & \mathbb{H}_{G_1^H}^T \end{pmatrix}$$

leads to the condition of Theorem 10.5. Furthermore, since the condition (10.12) holds only in the domain $\mathscr{D}_R$, the region of attraction includes the largest Lyapunov level set contained in $\mathscr{D}_R$. ■

Let us now consider the Lyapunov function in Case 2, together with a domain $\mathscr{D}_R$ defined as

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} \mathbb{H}_{G_0^H}^{-T} & 0 \\ 0 & \mathbb{P}_{G_1^P}^{-1} \end{pmatrix} \mathbb{R}_{G_0^R} \begin{pmatrix} \mathbb{H}_{G_0^H}^{-1} & 0 \\ 0 & \mathbb{P}_{G_1^P}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} > 0 \qquad (10.13)$$

For this case, the following result can be established:

Theorem 10.6 *The closed-loop system* (10.11) *is locally asymptotically stable in the domain $\mathscr{D}_S$, if there exist matrices $P_{\boldsymbol{i}_j^P} = P_{\boldsymbol{i}_j^P}^T$, $\boldsymbol{i}_j^P = pr_{G_j^P}^{\boldsymbol{i}}$, $H_{\boldsymbol{i}_j^H}$, $\boldsymbol{i}_j^H = pr_{G_j^H}^{\boldsymbol{i}}$, and $F_{\boldsymbol{i}_j^F}$, $\boldsymbol{i}_j^F = pr_{G_j^F}^{\boldsymbol{i}}$, $\boldsymbol{i} \in \mathbb{I}_{G_V}$, $j = 0, 1$, and $R_{\boldsymbol{i}_0^R} = R_{\boldsymbol{i}_0^R}^T$, $\boldsymbol{i}_0^R = pr_{G_0^R}^{\boldsymbol{i}}$, where $G_V = G_0^P \cup G_1^P \cup (G_0^M \oplus G_0^A) \cup (G_0^F \oplus G_0^B) \cup G_0^R$ so that*

$$\begin{pmatrix} -\mathbb{H}_{G_0^H} - \mathbb{H}_{G_0^H}^T + \mathbb{P}_{G_0^P} & (*) \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & -\mathbb{P}_{G_1^P} \end{pmatrix} + \mathbb{R}_{G_0^R} < 0$$

holds. Moreover, the region of attraction includes $\mathscr{D}_S$, where $\mathscr{D}_S$ is the largest Lyapunov level set included in $\mathscr{D}_R$.

Proof: The difference in the Lyapunov function is

$$\Delta V = \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -\mathbb{P}_{G_0^P}^{-1} & 0 \\ 0 & \mathbb{P}_{G_1^P}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}$$

and the closed-loop system can be expressed as

$$\begin{pmatrix} \mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} = 0$$

Furthermore, in the domain $\mathscr{D}_R$, (10.13) holds. Thus, in this domain, (10.11) is locally asymptotically stable, if there exists M so that

$$\begin{pmatrix} -\mathbb{P}_{G_0^P}^{-1} & 0 \\ 0 & \mathbb{P}_{G_1^P} \end{pmatrix} + M \begin{pmatrix} \mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} & -I \end{pmatrix} + (*)$$
$$+ \begin{pmatrix} \mathbb{H}_{G_0^H}^{-T} & 0 \\ 0 & \mathbb{P}_{G_1^P}^{-1} \end{pmatrix} \mathbb{R}_{G_0^R} \begin{pmatrix} \mathbb{H}_{G_0^H}^{-1} & 0 \\ 0 & \mathbb{P}_{G_1^P}^{-1} \end{pmatrix} < 0$$

Choosing

$$M = \begin{pmatrix} 0 \\ \mathbb{P}^{-1}_{G_1^P} \end{pmatrix}$$

and congruence with

$$\begin{pmatrix} \mathbb{H}^T_{G_0^H} & 0 \\ 0 & \mathbb{P}_{G_1^P} \end{pmatrix}$$

leads to

$$\begin{pmatrix} -\mathbb{H}^T_{G_0^H}\mathbb{P}^{-1}_{G_0^P}\mathbb{H}_{G_0^H} & (*) \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & -\mathbb{P}_{G_1^P} \end{pmatrix} + \mathbb{R}_{G_0^R} < 0$$

Applying Property 5 gives the condition of Theorem 10.6. Furthermore, since the condition (10.12) holds only in the domain $\mathscr{D}_R$, the region of attraction includes the largest Lyapunov level set contained in $\mathscr{D}_R$. ■

Let us now discuss the choice of the delays in G_0^P, G_0^H, G_0^F, and G_0^R. Similarly to the stability analysis case, since $\mathbb{R}_{G_0^R}$ is simply added, it should contain all the appearing indices. Furthermore, a reasonable choice is $G_0^H = G_0^F$, since they appear multiplied by A_z and B_z, respectively. Thus, G_V is now reduced to $G_V = G_0^P \cup G_1^P \cup (G_0^H \oplus G_0^A)$. Taking into account that G_0^H cannot contain positive indices (as they refer to future states) and following the reasoning in (Lendek et al., 2015), from the point of conservatism reduction approaches and reduced computational complexity, for Case 1, the best choices are $G_0^P = \{0, 0, \dots, 0\}$ and $G_0^F = G_0^H = \{0, 0, \dots, 0\}$ and for Case 2, $G_0^P = \{-1, -1, \dots, -1\}$ and $G_0^F = G_0^H = \{0, 0, \dots, 0, -1, \dots, -1\}$. These choices reduce the number of sums as follows

- Case 1: assuming $|G_0^P| = |G_0^H| = n_P$, the number of sums in Theorem 10.5 is $2n_P + 1$ and the number of LMIs to be solved (before relaxations) is r^{2n_p+1}.
- Case 2: assuming $|G_0^P| = |G_0^H| = 2n_P$, the number of sums in Theorem 10.6 is $2n_P + 1$ and the number of LMIs to be solved (before relaxations) is r^{2n_p+1}.

On the other hand, including negative, or even positive delays in G_0^R, although it increases the computational complexity, involves a condition on the trajectory of the states and thus it may increase the region of attraction. Furthermore, similarly to the stability analysis, it is possible to consider the future Lyapunov level sets included in future $\mathscr{D}_R$.

10.3.2 Example and discussion

In this section we illustrate the controller design results on a numerical example.

Example 10.2 Consider the nonlinear system:

$$\begin{aligned} x_1(k+1) &= 4\frac{x_1^2(k)}{1+x_1^2(k)} + 0.1x_2(k) \\ x_2(k+1) &= -x_1(k) - \frac{1}{2}x_2(k) + u(k) \end{aligned} \tag{10.14}$$

Using the sector nonlinearity approach (Ohtake et al., 2001), with x_1 as scheduling variable, the resulting TS model is

$$\boldsymbol{x}(k+1) = \sum_{i=1}^{2} h_i(\boldsymbol{z}(k)) \left(A_i\boldsymbol{x}(k) + Bu(k)\right)$$

with

$$A_1 = \begin{pmatrix} -2 & 0.1 \\ -1 & -2 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 2 & 0.1 \\ -1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Both local models are unstable and uncontrollable and controller design using either quadratic or nonquadratic Lyapunov functions results in unfeasible LMIs.

To illustrate local stabilization, consider the conditions of Theorem 10.5. For the different choices of the delays, the results are presented in Figure 10.4. In all the figures, the $*$ denotes the domain $\mathscr{D}_R$, the Lyapunov level sets are drawn, and the green o and x denote initial values from which trajectories converge to zero and diverge, respectively.

To circumvent this issue of the graphical representation with negative delays, the "predicted" domain and/or level sets are illustrated, as discussed in Section 9.3.

R_1: $G_0^P = \{0\}$, $G_0^H = G_0^F = \{0\}$, $G_0^R = \{0, 0, 1\}$, unstructured $\mathbb{R}_{G_0^R}$: as can be seen in Figure 10.4(a), the domain is reduced to zero. Note that even increasing the number of indices used, e.g., $G_0^P = \{0, 0, 0, 0\}$, $G_0^H = G_0^F = \{0, 0, 0, 0\}$, $G_0^R = \{0, 0, 0, 0, 1, 1, 1, 1, 1\}$, the domain does not increase.

Since an increase of the domain cannot be obtained using a full $\mathbb{R}_{G_0^R}$, in what follows, a block-diagonal one is considered.

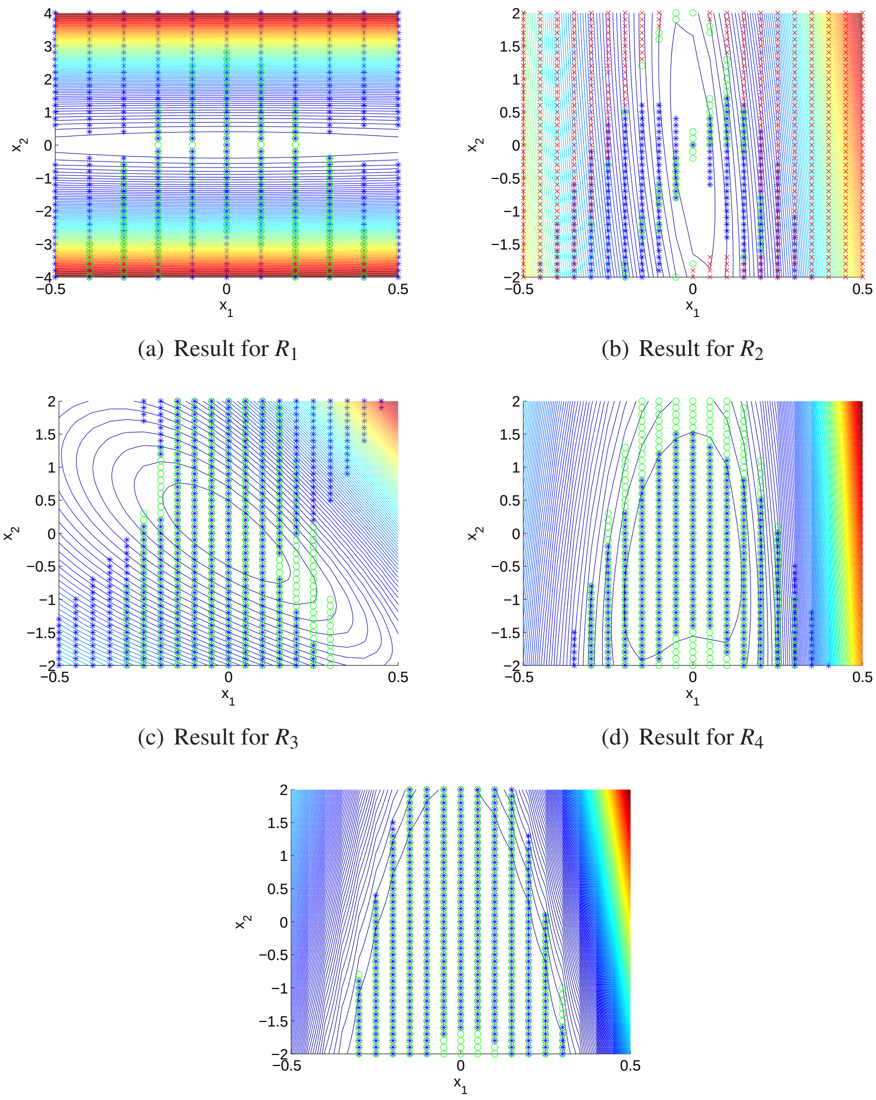
R_2: $G_0^P = \{0, 0\}$, $G_0^H = G_0^F = \{0\}$, $G_0^R = \{0, 0, 1, 1\}$: as can be seen in Figure 10.4(b), the domain is still reduced to zero. A similar result is obtained for the classic choices, i.e., $G_0^P = \{0\}$, $G_0^H = G_0^F = \{0\}$, $G_0^R = \{0, 0, 1\}$.

R_3: $G_0^P = \{0, 0, 0\}$, $G_0^H = G_0^F = \{0, 0\}$, $G_0^R = \{0, 0, 0, 1, 1, 1\}$: as can be seen in Figure 10.4(c), the domain has increased. This can be somewhat, but not significantly increased extending the number of indices used.

Therefore, in what follows, we consider including past samples in the domain definition.

R_4: $G_0^P = \{0,0,0\}$, $G_0^H = G_0^F = \{0,0\}$, $G_0^R = \{-1,0,0,0,1,1,1\}$: as can be seen in Figure 10.4(d), the domain increases. Note that including the delayed indices in G_0^P does not have any positive effect on the domain.

R_5: $G_0^P = \{0,0,0\}$, $G_0^H = G_0^F = \{0,0\}$, $G_0^R = \{-2,-1,0,0,0,1,1,1\}$: as can be seen in Figure 10.4(e), the domain further increases.



(a) Result for R_1

(b) Result for R_2

(c) Result for R_3

(d) Result for R_4

(e) Result for R_5

Figure 10.4: Results for Example 10.2

□

Similarly to the stability problem, by introducing further delays in the domain matrix, the domain is increased. Furthermore, the "predicted" Lyapunov level sets asymptotically converge to the domain where the system is stabilized.

It has to be noted that using delays in $\mathbb{H}_{G_0^H}$ and $\mathbb{F}_{G_0^F}$ requires the specification of $\boldsymbol{x}(-1)$, $\boldsymbol{x}(-2)$, etc. This means that the region recovered will depend not only on the initial states, but also on these delayed states. This is in particular the case of Theorem 10.6, where, in order to obtain less conservative results, G_0^F and G_0^H should include -1.

Similarly to stability analysis, a structured $\mathbb{R}_{G_0^R}$ seems to lead to an increase of the region. However, one should keep in mind that the region is actually determined not only by $\mathbb{R}_{G_0^R}$, but also by $\mathbb{H}_{G_0^H}$ and $\mathbb{R}_{G_0^R}$. The elucidation of the problem of how exactly the structure of $\mathbb{R}_{G_0^R}$ should be chosen such that the region is optimized is left for future research.

10.4 Extension: local set point tracking control

Consider now the discrete-time TS system

$$\begin{aligned} \boldsymbol{x}(k+1) &= A_z\boldsymbol{x}(k) + B_z\boldsymbol{u}(k) \\ \boldsymbol{y}(k) &= C\boldsymbol{x}(k) \end{aligned} \tag{10.15}$$

defined on a domain $\mathscr{D}$ including the origin, where $\boldsymbol{y}$ denotes the output.

Our goal is to determine a control law, such that $\boldsymbol{y}$ tracks a desired reference signal $\boldsymbol{y}_r$. For this, we will use the auxiliary state variable $\boldsymbol{x}^I(k)$ – corresponding to an integral term – with the dynamics given by $\boldsymbol{x}^I(k+1) = \boldsymbol{x}^I(k) - C\boldsymbol{x}(k) + \boldsymbol{y}_r$. The extended state dynamics are

$$\begin{aligned} \boldsymbol{x}(k+1) &= A_z\boldsymbol{x}(k) + B_z\boldsymbol{u}(k) \\ \boldsymbol{x}^I(k+1) &= \boldsymbol{x}^I(k) - C\boldsymbol{x}(k) + \boldsymbol{y}_r \end{aligned}$$

Denoting $\boldsymbol{x}^e(k) = \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}^I(k) \end{pmatrix}$, $A_z^e = \begin{pmatrix} A_z & 0 \\ -C & I \end{pmatrix}$, $B_z^e = \begin{pmatrix} B_z \\ 0 \end{pmatrix}$, $D = \begin{pmatrix} 0 \\ I \end{pmatrix}$ we have

$$\boldsymbol{x}^e(k+1) = A_z^e\boldsymbol{x}^e(k) + B_z^e\boldsymbol{u}(k) + D\boldsymbol{y}_r$$

and we consider a control law $\boldsymbol{u}(k) = -F_zH_z^{-1}\boldsymbol{x}^e(k)$. The closed-loop system is

$$\boldsymbol{x}^e(k+1) = (A_z^e - B_z^eF_zH_z^{-1})\boldsymbol{x}^e(k) + D\boldsymbol{y}_r \tag{10.16}$$

If the closed-loop system (10.16) with $\boldsymbol{y}_r = 0$ is globally uniformly asymptotically stable, then it is also input-to-state stable (ISS) with respect to the exogenous input $\boldsymbol{y}_r$. In what follows, we determine conditions for the local tracking and determine a region where tracking is possible.

Similarly to the stabilization problem, consider the nonquadratic Lyapunov function $V = \boldsymbol{x}^{eT}(k)P^{-1}\boldsymbol{x}^e(k)$, together with the constraint $\begin{pmatrix} \boldsymbol{x}^e(k) \\ \boldsymbol{x}^e(k+1) \end{pmatrix}^T R \begin{pmatrix} \boldsymbol{x}^e(k) \\ \boldsymbol{x}^e(k+1) \end{pmatrix}$ between consecutive samples. We have the result:

Theorem 10.7 *The discrete-time nonlinear model* (10.16) *is locally ISS with respect to the exogenous input $\boldsymbol{y}_r$ in the domain $\mathscr{D}_S$, if there exist P, F_z, H_z, and W, so that*

$$\begin{pmatrix} -H_z^T - H_z + P & (*) \\ A_z^e H_z - B_z^e F_z & -P_+ \end{pmatrix} + W < 0 \tag{10.17}$$

where $\mathscr{D}_R$ is given by

$$\mathscr{D}_R = \left\{ \boldsymbol{x}^e(k) \in \mathscr{D} \,|\, \begin{pmatrix} \boldsymbol{x}^e(k) \\ \boldsymbol{x}^e(k+1) \end{pmatrix}^T \begin{pmatrix} H_z^{-T} & 0 \\ 0 & P_+^{-1} \end{pmatrix} W \begin{pmatrix} H_z^{-1} & 0 \\ 0 & P_+^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}^e(k) \\ \boldsymbol{x}^e(k+1) \end{pmatrix} \geq 0 \right\}$$

and $\mathscr{D}_S$ is the largest Lyapunov level set included in $\mathscr{D}_R \bigcap \mathscr{D}$.

Proof: Consider the candidate Lyapunov function $V = (\boldsymbol{x}^e)^T(k)P^{-1}\boldsymbol{x}^e(k)$. The difference in V is

$$\begin{aligned} \Delta V &= (\boldsymbol{x}^e)^T(k+1)P_+^{-1}\boldsymbol{x}^e(k+1) - (\boldsymbol{x}^e)^T(k)P^{-1}\boldsymbol{x}^e(k) \\ &= \begin{pmatrix} \boldsymbol{x}^e(k) \\ \boldsymbol{x}^e(k+1) \end{pmatrix}^T \begin{pmatrix} -P^{-1} & 0 \\ 0 & P_+^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}^e(k) \\ \boldsymbol{x}^e(k+1) \end{pmatrix} \end{aligned}$$

The dynamics (10.16) can be written as

$$\begin{pmatrix} A_z^e - B_z^e F_z H_z^{-1} & -I & D \end{pmatrix} \begin{pmatrix} \boldsymbol{x}^e(k) \\ \boldsymbol{x}^e(k+1) \\ \boldsymbol{y}_r \end{pmatrix} = 0$$

Then, we have

$$\begin{aligned} \Delta V = &\begin{pmatrix} \boldsymbol{x}^e(k) \\ \boldsymbol{x}^e(k+1) \end{pmatrix}^T \begin{pmatrix} -P^{-1} & 0 \\ 0 & P_+^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}^e(k) \\ \boldsymbol{x}^e(k+1) \end{pmatrix} \\ &+ \begin{pmatrix} \boldsymbol{x}^e(k) \\ \boldsymbol{x}^e(k+1) \\ \boldsymbol{y}_r \end{pmatrix}^T \left(\begin{pmatrix} 0 \\ P_+^{-1} \\ 0 \end{pmatrix} \begin{pmatrix} A_z^e - B_z^e F_z H_z^{-1} & -I & D \end{pmatrix} + (*) \right) \begin{pmatrix} \boldsymbol{x}^e(k) \\ \boldsymbol{x}^e(k+1) \\ \boldsymbol{y}_r \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{x}^e(k) \\ \boldsymbol{x}^e(k+1) \end{pmatrix}^T \begin{pmatrix} -P^{-1} & (*) \\ P_+^{-1}(A_z^e - B_z^e F_z H_z^{-1}) & -P_+^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}^e(k) \\ \boldsymbol{x}^e(k+1) \end{pmatrix} \\ &+ 2\boldsymbol{y}_r P_+^{-1} \left((A_z^e - B_z^e F_z H_z^{-1})\boldsymbol{x}^e(k) + D\boldsymbol{y}_r \right) \\ &\leq \mathscr{Q}(\boldsymbol{x}^e) + 2\delta \|\boldsymbol{y}_r\| \|\boldsymbol{x}^e(k)\| + 2\alpha \|\boldsymbol{y}_r\|^2 \end{aligned}$$

where

$$\mathscr{Q}(\boldsymbol{x}^e) = \begin{pmatrix} \boldsymbol{x}^e(k) \\ \boldsymbol{x}^e(k+1) \end{pmatrix}^T \begin{pmatrix} -P^{-1} & (*) \\ P_+^{-1}(A_z^e - B_z^e F_z H_z^{-1}) & -P_+^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}^e(k) \\ \boldsymbol{x}^e(k+1) \end{pmatrix}$$

and $\delta \geq 0$ and $\alpha \geq 0$ are bounding constants, i.e., $\|P_+^{-1}(A_z - B_z F_z H_z^{-1})\| \leq \delta$ and $\|P_+^{-1}D\| \leq \alpha$.

Let us now consider $\mathscr{Q}(\boldsymbol{x}^e)$. Assuming that there exists R and a domain $\mathscr{D}_R$ such that

$$\begin{pmatrix} \boldsymbol{x}^e(k) \\ \boldsymbol{x}^e(k+1) \end{pmatrix}^T R \begin{pmatrix} \boldsymbol{x}^e(k) \\ \boldsymbol{x}^e(k+1) \end{pmatrix} \geq 0$$

$\mathscr{Q}(\boldsymbol{x}^e) < 0$, if

$$\begin{pmatrix} -P^{-1} & (*) \\ P_+^{-1}(A_z^e - B_z^e F_z H_z^{-1}) & -P_+^{-1} \end{pmatrix} + R < 0$$

Congruence with $\begin{pmatrix} H_z^T & 0 \\ 0 & P_+ \end{pmatrix}$ and applying Proposition 5 gives

$$\begin{pmatrix} -H_z^T - H_z + P & (*) \\ A_z^e H_z - B_z^e F_z & -P_+ \end{pmatrix} + \begin{pmatrix} H_z^T & 0 \\ 0 & P_+ \end{pmatrix} R \begin{pmatrix} H_z & 0 \\ 0 & P_+ \end{pmatrix} < 0 \tag{10.18}$$

or

$$\begin{pmatrix} -H_z^T - H_z + P & (*) \\ A_z^e H_z - B_z^e F_z & -P_+ \end{pmatrix} + W < 0$$

Furthermore in $\mathscr{D}_R$ where (10.18) holds, $\exists \lambda > 0$ so that $\mathscr{Q}(\boldsymbol{x}^e) < -\lambda \|\boldsymbol{x}^e(k)\|^2$. Consequently, in this domain,

$$\Delta V \leq -\lambda \|\boldsymbol{x}^e(k)\|^2 + 2\delta \|\boldsymbol{y}_r\| \|\boldsymbol{x}^e(k)\| + 2\alpha \|\boldsymbol{y}_r\|^2$$

and by the completion of squares, $2\delta \|\boldsymbol{y}_r\| \|\boldsymbol{x}^e(k)\| \leq \frac{1}{\theta} \|\boldsymbol{x}^e(k)\|^2 + \delta^2 \theta \|\boldsymbol{y}_r\|^2$, $\forall \theta > 0$, thus

$$\Delta V \leq -\lambda \|\boldsymbol{x}^e(k)\|^2 + \frac{1}{\theta} \|\boldsymbol{x}^e(k)\|^2 + \delta^2 \theta \|\boldsymbol{y}_r\|^2 + 2\alpha \|\boldsymbol{y}_r\|^2$$

Choosing $\theta > \frac{1}{\lambda}$ and denoting $c_1 = \lambda - \frac{1}{\theta} > 0$ and $c_2 = \delta^2 \theta + 2\alpha$, we have

$$\Delta V \leq -c_1 \|\boldsymbol{x}^e(k)\|^2 + c_2 \|\boldsymbol{y}_r\|^2$$

Furthermore, consider $\tau \in (0, 1)$. Then,

$$\begin{aligned} \Delta V &\leq -(1-\tau) c_1 \|\boldsymbol{x}^e(k)\|^2 - \tau c_1 \|\boldsymbol{x}^e(k)\|^2 + c_2 \|\boldsymbol{y}_r\|^2 \\ &\leq -(1-\tau) c_1 \|\boldsymbol{x}^e(k)\|^2 \qquad \forall \|\boldsymbol{x}^e\|^2 \geq \frac{c_2}{\tau c_1} \|\boldsymbol{y}_r\|^2 \end{aligned}$$

i.e., the closed-loop system (10.16) is ISS with respect to the exogenous input $\boldsymbol{y}_r$, with an ultimate bound given by $\frac{c_2}{\tau c_1}$. ■

While the bounding constants α and δ do not affect the feasibility of the developed LMI conditions, they do affect the bound above and how closely the systems output $\boldsymbol{y}$ will track the reference signal $\boldsymbol{y}_r$. Thus, in what follows, we develop LMI conditions for the minimization of this bound. Recall that $\|P_+^{-1}(A_z^e - B_z^e F_z H_z^{-1})\| \leq \delta$ and $\|P_+^{-1}D\| \leq \alpha$ and consider first $\|P_+^{-1}D\| \leq \alpha$. This is satisfied if

$$D^T P_+^{-1} P_+^{-1} D \leq \alpha^2 I$$
$$\alpha^2 I - D^T P_+^{-1} P_+^{-1} D \geq 0$$

By the Schur complement, we have

$$\begin{pmatrix} \alpha^2 I & D^T P_+^{-1} \\ P_+^{-1} D & I \end{pmatrix} \geq 0$$

Congruence with $\mathrm{diag}(I\,P_+)$ gives

$$\begin{pmatrix} \alpha^2 I & D^T \\ D & P_+ P_+ \end{pmatrix} \geq 0$$

and using Proposition 5 on $P_+ P_+$ we obtain

$$\begin{pmatrix} \alpha^2 I & D^T \\ D & 2P_+ - I \end{pmatrix} \geq 0 \tag{10.19}$$

Consider now $\|P_+^{-1}(A_z^e - B_z^e F_z H_z^{-1})\| \leq \delta$. This is satisfied, if

$$(P_+^{-1}(A_z^e - B_z^e F_z H_z^{-1}))^T P_+^{-1}(A_z^e - B_z^e F_z H_z^{-1}) \leq \delta^2 I$$

Using the Schur complement and congruence with $\mathrm{diag}(H_z^T\,P_+)$ gives

$$\begin{pmatrix} H_z^T \delta^2 H_z & (*) \\ A_z^e H_z - B_z^e F_z & P_+ P_+ \end{pmatrix} \geq 0$$

and using Proposition 5 on both $H_z^T \delta^2 H_z$ and $P_+ P_+$ leads to

$$\begin{pmatrix} H_z^T + H_z - \frac{1}{\delta^2} I & (*) \\ A_z^e H_z - B_z^e F_z & 2P_+ - I \end{pmatrix} \geq 0 \tag{10.20}$$

Furthermore, since the bound $\frac{c_2}{\tau c_1}$ depends on c_1, which in turn depends on λ, λ should be maximized. Then, $\mathscr{Q}(\boldsymbol{x}^e) \leq -\lambda \|\boldsymbol{x}^e(k)\|$, if

$$\begin{pmatrix} -P^{-1} + \lambda & (*) \\ P_+^{-1}(A_z^e - B_z^e F_z H_z^{-1}) & -P_+^{-1} \end{pmatrix} + R < 0$$

Similarly to the proof of Theorem 10.7, congruence with $\begin{pmatrix} H_z^T & 0 \\ 0 & P_+ \end{pmatrix}$ and applying Proposition 5 gives

$$\begin{pmatrix} -H_z^T - H_z + \lambda H_z^T H_z + P & (*) \\ A_z^e H_z - B_z^e F_z & -P_+ \end{pmatrix} + W < 0$$

which, after applying the Schur complement results in

$$\begin{pmatrix} -H_z^T - H_z + P & (*) & (*) \\ A_z^e H_z - B_z^e F_z & -P_+ & 0 \\ H_z^T & 0 & -\frac{1}{\lambda} I \end{pmatrix} + \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} < 0 \tag{10.21}$$

This result can be summarized as follows.

Corollary 10.8 *The discrete-time nonlinear model* (10.16) *is locally ISS with respect to the exogenous input* $\mathbf{y}_r$ *in the domain* $\mathscr{D}_S$*, if there exist* P*,* F_z*,* H_z*, and* W*, so that condition* (10.17) *holds. Furthermore, the ultimate bound on the states can be minimized by solving*

minimize α*,* δ*, maximize* λ

subject to (10.19), (10.20), (10.21)

together with (10.17).

10.5 Conclusions

This chapter presented conditions for local stabilization of a TS fuzzy model. First a common quadratic Lyapunov function has been used and then the results have been extended to nonquadratic Lyapunov functions. The main idea was to combine the search for the control gains with the maximization of the domain of attraction using an LMI formalism. The developed conditions have also been generalized using delayed nonquadratic Lyapunov functions and controller gains and have been extended to local set point tracking control. The conditions have been illustrated on numerical examples.

Chapter 11

Observer design

This chapter considers local observer design for system (8.4), repeated here for convenience:

$$\begin{aligned} \boldsymbol{x}(k+1) &= A_z \boldsymbol{x}(k) + B_z \boldsymbol{u}(k) \\ \boldsymbol{y}(k) &= C_z \boldsymbol{x}(k) \end{aligned} \tag{11.1}$$

defined on the domain $\mathscr{D}$ including the origin.

Our objective is to design a state observer of the form

$$\begin{aligned} \hat{\boldsymbol{x}}(k+1) &= A_z \hat{\boldsymbol{x}}(k) + B_z \boldsymbol{u}(k) + M_z^{-1} L_z (\boldsymbol{y}(k) - \hat{\boldsymbol{y}}(k)) \\ \hat{\boldsymbol{y}}(k) &= C_z \hat{\boldsymbol{x}}(k) + D_z \boldsymbol{u}(k) \end{aligned} \tag{11.2}$$

and the goal is then to find the observer gains M_z and L_z such that the state estimate $\widehat{\boldsymbol{x}}$ asymptotically converges towards the system state $\boldsymbol{x}$, which is equivalent to ensuring the convergence of the estimation error to zero. Under the assumption that the scheduling vector depends only on measured variables, the estimation error dynamics, from (11.1) and (11.2), are given by

$$\boldsymbol{e}(k+1) = (A_z - M_z^{-1} L_z C_z) \boldsymbol{e}(k) \tag{11.3}$$

Similarly to stability analysis and stabilization, we first consider a quadratic Lyapunov function and then the results are generalized to nonquadratic Lyapunov functions. Together with the observer gains, a domain of attraction of the error dynamics is also obtained. The presented approaches are discussed and illustrated on numerical examples.

11.1 Quadratic design

We first present results obtained by using a quadratic Lyapunov function $V = \boldsymbol{e}(k)^T P \boldsymbol{e}(k)$.

11.1.1 Design conditions

Using a quadratic Lyapunov function, the following result can be established.

Theorem 11.1 *There exists an observer of the form* (11.2) *for the system* (11.1)*, such that the state estimation error* $\boldsymbol{e}(k)$ *asymptotically converges towards the origin in the largest Lyapunov level set included in the domain* $\mathscr{D} \cap \mathscr{D}_R$*, if there exists a positive scalar* ε*, a symmetric positive definite matrix* $P \in \mathbb{R}^{n\times n}$*, a symmetric matrix* $R \in \mathbb{R}^{2n\times 2n}$*, matrices* $M_i \in \mathbb{R}^{n\times n}$ *and* $L_i \in \mathbb{R}^{n\times n_y}$*, for* $i \in \mathscr{I}_r$*, satisfying the following matrix inequality*

$$\begin{pmatrix} \varepsilon(M_z A_z - L_z C_z) + (*) - P & (*) \\ M_z A_z - L_z C_z - \varepsilon M_z^T & P - M_z - (*) \end{pmatrix} + R < 0 \tag{11.4}$$

Proof: In order to prove the local asymptotic convergence of the state estimation error, we have to prove the negativeness of the difference of the quadratic Lyapunov function, namely

$$\Delta V(k) = \begin{pmatrix} e(k) \\ e(k+1) \end{pmatrix}^T \begin{pmatrix} -P & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} e(k) \\ e(k+1) \end{pmatrix} < 0 \tag{11.5}$$

in $\mathscr{D}_R$ and along the trajectory of (11.3). The dynamics of the state estimation error (11.3) can also be written as the following equality constraint

$$\begin{pmatrix} A_z - M_z^{-1} L_z C_z & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k+1) \end{pmatrix} = 0 \tag{11.6}$$

Defining M by

$$M = \begin{pmatrix} \varepsilon M_z \\ M_z \end{pmatrix} \tag{11.7}$$

the inequality (11.4) can be written as

$$R + \begin{pmatrix} -P & 0 \\ 0 & P \end{pmatrix} + M \begin{pmatrix} A_z - M_z^{-1} L_z C_z & -I \end{pmatrix} + (*) < 0 \tag{11.8}$$

Pre- and post-multiplying (11.8) by $\boldsymbol{v}^T = (\boldsymbol{e}^T(k)\ \boldsymbol{e}^T(k+1))$, from Lemma 2.13, it implies that

$$\begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k+1) \end{pmatrix}^T \left(\begin{pmatrix} P & 0 \\ 0 & -P \end{pmatrix} - R \right) \begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k+1) \end{pmatrix} > 0 \tag{11.9}$$

holds along the trajectory of (11.3). From Property 3, with $\tau = 1$, $F_1 = R$ and $F_0 = diag(P, -P)$, inequality (11.9) implies that, in $\mathscr{D}_R$ (i.e. if $z^T R z > 0$ is satisfied), we have

$$z^T diag(P, -P) z > 0 \tag{11.10}$$

This last inequality is equivalent to (11.5) and achieves the proof. ∎

Since the matrix inequality (11.4) is based on a double summation both involving the same weighting functions $h_i(z(k))$, the relaxation schemes in Lemmas 2.10 or 2.11 can easily be used to derive sufficient conditions that are numerically tractable with classical dedicated softwares. Moreover, in this case, the observer gains in (11.2) are directly obtained from the numerical solution of the LMI problem given in the following corollary.

Corollary 11.2 *There exists an observer* (11.2) *for the system* (11.1)*, such that the state estimation error $\boldsymbol{e}(k)$ asymptotically converges towards the origin in the largest Lyapunov level set included in $\mathscr{D} \cap \mathscr{D}_R$, if there exists a positive scalar ε, a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, a symmetric matrix $R \in \mathbb{R}^{2n\times 2n}$, matrices $M_i \in \mathbb{R}^{n\times n}$ for $i \in \mathscr{I}_r$ and $L_i \in \mathbb{R}^{n\times n_y}$ satisfying the conditions of Lemmas 2.10 or 2.11 where Γ_{ij}, for $(i,j) \in \mathscr{I}_r^2$, are defined as*

$$\Gamma_{ij} = \begin{pmatrix} \varepsilon(M_jA_i - L_iC_j) + (*) - P & (*) \\ M_jA_i - L_iC_j - \varepsilon M_j^T & P - M_j - (*) \end{pmatrix} + R \qquad (11.11)$$

Remark: Due to ε, the conditions in Corollary 11.2 are parameterized LMIs. To obtain LMIs, ε may be fixed or can be searched for in a logarithmically spaced family of values (Oliveira and Peres, 2007), $\varepsilon \in \{10^{-6}, 10^{-5}, \ldots, 10^{6}\}$.

11.1.2 Example and discussion

In what follows, we illustrate the developed conditions on a numerical example.

Example 11.1 Consider the four-rule TS model defined on the domain $\mathscr{D} = \{x_1, x_2 \in [-2,2]\}$, with local matrices

$$A_1 = \begin{pmatrix} -2 & -1 \\ 1 & 4.25 \end{pmatrix} \qquad C_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} -2 & 1 \\ 1 & -4.25 \end{pmatrix} \qquad C_2 = \begin{pmatrix} -5 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 2 & -1 \\ 1 & 4.25 \end{pmatrix} \qquad C_3 = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 2 & 1 \\ 1 & -4.25 \end{pmatrix} \qquad C_4 = \begin{pmatrix} -5 & 0 \end{pmatrix}$$

and membership functions

$$\begin{aligned} h_1(x_1) &= \frac{(2-x_1)}{4}\frac{(1-\sin(x_1))}{2} \\ h_2(x_1) &= \frac{(2-x_1)}{4}\frac{(1+\sin(x_1))}{2} \\ h_3(x_1) &= \frac{(2+x_1)}{4}\frac{(1-\sin(x_1))}{2} \\ h_4(x_1) &= \frac{(2+x_1)}{4}\frac{(1+\sin(x_1))}{2} \end{aligned}$$

The above membership functions have the convex sum property.

Note that for this specific TS model, neither a PDC nor a non-PDC observer can be designed, independent of the Lyapunov function used (common quadratic or nonquadratic) – the LMI conditions are unfeasible. In what follows, we study if a local observer can be designed using the conditions of Theorem 11.1. In order to maximize the domain $\mathscr{D}_R$, the matrix R has been chosen with the following structure

$$R = diag(\bar{R}, -I) \tag{11.12}$$

where $\bar{R}$ is a decision variable whose trace is being maximized.

The conditions of Theorem 11.1 with $\varepsilon = 0$ and using Lemma 2.10 for relaxation are feasible and the following results have been obtained:

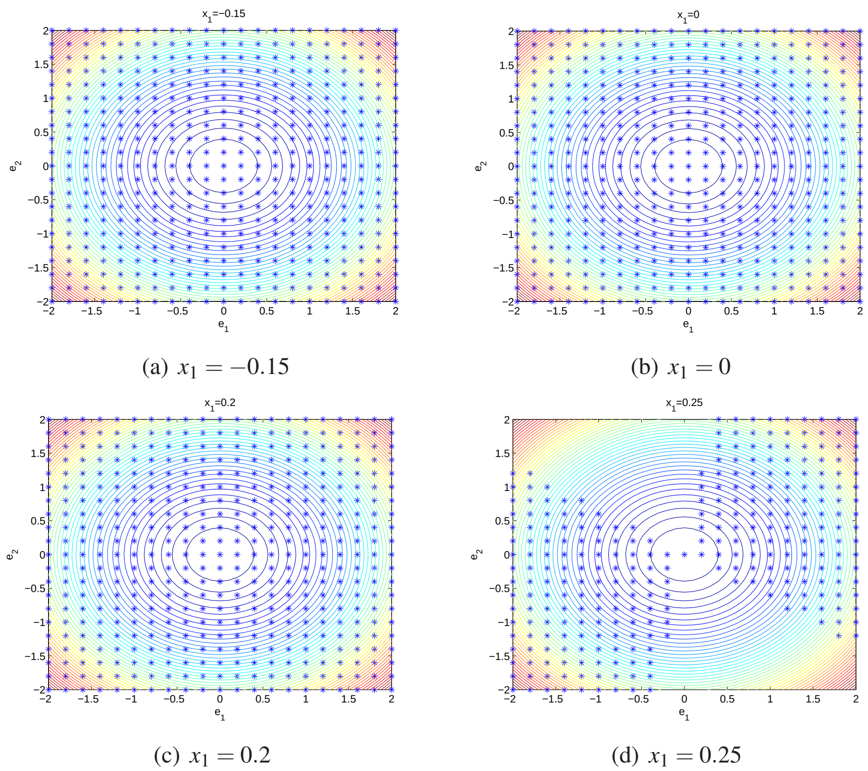
$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\begin{aligned} M_1 &= 10^{-3}\begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} & L_1 &= 10^{-3}\begin{pmatrix} 0.1351 \\ 0.3273 \end{pmatrix} \\ M_2 &= 10^{-3}\begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} & L_2 &= 10^{-3}\begin{pmatrix} 0.0027 \\ -0.2136 \end{pmatrix} \\ M_3 &= 10^{-3}\begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} & L_3 &= 10^{-4}\begin{pmatrix} 0.1377 \\ -0.6871 \end{pmatrix} \\ M_4 &= 10^{-3}\begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} & L_4 &= 10^{-3}\begin{pmatrix} 0.0316 \\ -0.1324 \end{pmatrix} \end{aligned}$$

Even if the Lyapunov function in this case is quadratic and thus does not depend on the scheduling variables, one should note that due to its definition, $\mathscr{D}_R$ depends both on $\boldsymbol{e}(k)$ and $\boldsymbol{e}(k+1)$ and thus indirectly depends on the scheduling variable $\boldsymbol{z}(k)$. Consequently, the actual region of attraction $\mathscr{D}_R$ depends on $\boldsymbol{z}(k)$.

To verify in which domain the estimation error converges to zero, the domain $\mathscr{D}_R$ and the Lyapunov level sets are presented in Figure 11.1. Note that for $x_1 \leq -0.2$

and $x_1 \geq 0.25$, the Lyapunov level sets included in $\mathscr{D}_R$ are reduced to zero, as seen in Figure 11.1(d). A trajectory of the error dynamics, with $\boldsymbol{x}(0) = [-0.15, -1]^T$ and $\boldsymbol{e}(0) = [-1, 1]^T$ is presented in Figure 11.2.



(a) $x_1 = -0.15$ (b) $x_1 = 0$

(c) $x_1 = 0.2$ (d) $x_1 = 0.25$

Figure 11.1: The domain $\mathscr{D}_R$ (*) and the Lyapunov level sets for different values of the scheduling variables.

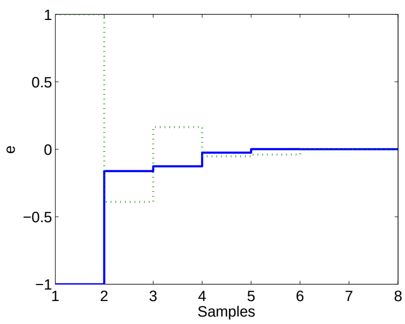


Figure 11.2: A trajectory of the estimation error.

□

11.2 Non-quadratic design

In order to obtain a more general result, a nonquadratic Lyapunov function is exploited here and the following theorem details the existence condition of the observer (11.2) in the non-quadratic case.

Theorem 11.3 *There exists an observer* (11.2) *for the system* (11.1)*, such that the state estimation error* $\boldsymbol{e}(k)$ *asymptotically converges towards the origin in the largest Lyapunov level set included in* $\mathscr{D} \cap \mathscr{D}_R$*, if there exists a positive scalar* ε*, symmetric positive definite matrices* $P_i \in \mathbb{R}^{n\times n}$*, a symmetric matrix* $R \in \mathbb{R}^{2n\times 2n}$*, matrices* $M_i \in \mathbb{R}^{n\times n}$ *and* $L_i \in \mathbb{R}^{n\times n_y}$*, for* $i \in \mathscr{I}_r$*, satisfying the following inequalities.*

$$\begin{pmatrix} \varepsilon(M_zA_z - L_zC_z) + (*) - P_z & (*) \\ M_zA_z - L_zC_z - \varepsilon M_z^T & P_{z+} - M_z - (*) \end{pmatrix} + R < 0 \tag{11.13}$$

Proof: Consider the Lyapunov function $V = \boldsymbol{e}(k)^T P_z \boldsymbol{e}(k)$. The proof follows the same line as the one of the theorem 11.1, except that the difference of the Lyapunov function is given by

$$\Delta V = \begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k+1) \end{pmatrix}^T \begin{pmatrix} -P_z & 0 \\ 0 & P_{z+} \end{pmatrix} \begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k+1) \end{pmatrix} \tag{11.14}$$

■

Sufficient parameterized LMI conditions for Theorem 11.3 can be set up as follows.

Corollary 11.4 *There exists an observer* (11.2) *for the system* (11.1)*, such that the state estimation error* $\boldsymbol{e}(k)$ *asymptotically converges towards the origin in the largest Lyapunov level set included in* $\mathscr{D} \cap \mathscr{D}_R$*, if there exists a positive scalar* ε*, symmetric positive definite matrices* $P_i \in \mathbb{R}^{n\times n}$*, a symmetric matrix* $R \in \mathbb{R}^{2n\times 2n}$*, matrices* $M_i \in \mathbb{R}^{n\times n}$ *for* $i \in \mathscr{I}_r$ *and* $L_i \in \mathbb{R}^{n\times n_y}$ *satisfying the conditions of Lemmas 2.10 or 2.11 with*

$$\Gamma_{ijk} = \begin{pmatrix} \varepsilon(M_jA_i - L_iC_j) + (*) - P_j & (*) \\ M_jA_i - L_iC_j - \varepsilon M_j^T & P_k - M_j - (*) \end{pmatrix} + R \tag{11.15}$$

Remark: For different ε, different R matrices, but at the same time, different observer gains can be obtained. Since the error dynamics will be different for each case, comparing the corresponding domains is far from trivial.

11.3 Conclusions

In this chapter, LMI conditions have been proposed for the design of local state observers for TS fuzzy systems. Both quadratic and non-quadratic Lyapunov functions have been employed to derive the conditions. It has to be noted that although the conditions of Theorems 11.1 and 11.3 may be used to design local observers, the shortcoming of this design is that the domain has to be verified a posteriori. This design can naturally be extended for α sample variation of the Lyapunov function and possible improvement may be obtained with the introduction of delayed Lyapunov functions. However, the domain, unlike in case of stability analysis and stabilization, naturally depends not only on the estimation error, but also on the states of the system and the estimated states, thus making the verification cumbersome. This problem may be alleviated by using a domain description depending on the measurement error instead of the estimation error.

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Part IV

Other topics and future plans

Chapter 12

Other directions

In addition to my main work thread on periodic and switching systems and local analysis and design, I have also worked together with students on offshoot directions from my research: descriptor systems, and applications in diverse fields. These directions are briefly overviewed next, in Sections 12.1 and 12.2, respectively.

12.1 Descriptor systems

The dynamic models of mechanical systems are generally determined using the Euler-Lagrange equations (Lewis et al., 2004; Spong et al., 2005), which give second-order vector differential equations. Once the Euler-Lagrange equations are obtained, the state-space representation is naturally in a descriptor form (Lewis et al., 2004; Luenberger, 1977; Chen et al., 2009), i.e., it has the mass matrix on the left-hand side. TS descriptor models have been introduced in (Taniguchi et al., 1999). Next to generalizing the standard TS model, they allow obtaining a smaller number of conditions (Taniguchi et al., 2000; Guerra and Vermeiren, 2004; Guerra et al., 2007) by keeping apart the nonlinearities on the two sides of the equation. A major direction of our research here has been the investigation with PhD student V. Estrada-Manzo of such descriptor systems. First, we considered continuous-time descriptor systems and using Finsler's lemma to decouple the control law from the Lyapunov function, an improved controller design method has been proposed in (P27). For observer design, we have proposed a pure LMI formulation of the conditions in (P26), a major step forward from existing BMI conditions. A generalization of this method has been presented in (P18). We have also proposed an unknown input observer design method in (P19), where the conditions are pure LMIs.

Next, we investigated discrete-time descriptor systems. Two approaches for controller design via two different Lyapunov functions have been proposed in (P25). The design methods have been generalized in (P17). An observer design method, using Finsler's lemma to decouple the Lyapunov matrices and the observer gains has been presented in (P24). This approach allows more general observer structures and thus a larger number of decision variables and has been generalized in (P16). An improvement of this method for the quadratic case has been illustrated in (P21). Next, we have developed a static output feedback control method for the case when there are several output matrices.

Inspired by the way the Lyapunov function and the controller/observer gains can be decoupled in the discrete-time case, we have developed a similar approach for the continuous case, for static output feedback control, in (P20).

The following publications were discussed in this section:

(P16) V. Estrada-Manzo, T. M. Guerra, Zs. Lendek, Generalized observer design for discrete-time T-S descriptor models. Neurocomputing, vol. 182, pages 210-220, 2016.

(P17) V. Estrada-Manzo, Zs. Lendek, T. M. Guerra, Ph. Pudlo, Controller design for discrete-time descriptor models: a systematic LMI approach. IEEE Transactions on Fuzzy Systems, vol. 23 , no. 5, pages 1608-1621, 2015.

(P18) T. M. Guerra, V. Estrada-Manzo, Zs. Lendek, Observer design for Takagi-Sugeno descriptor models: an LMI approach. Automatica, vol. 52 , no. 2, pages 154-159, 2015.

(P19) V. Estrada-Manzo, Zs. Lendek, T. M. Guerra, Unknown input estimation for nonlinear descriptor systems via LMIs and Takagi-Sugeno models. In Proceedings of the 54th IEEE Conference on Decision and Control, pages 1-6, Osaka, Japan, December 2015.

(P20) V. Estrada-Manzo, T. M. Guerra, Zs. Lendek, Static output feedback control for continuous-time TS descriptor models: decoupling the Lyapunov function. In Proceedings of the 2015 IEEE International Conference on Fuzzy Systems, pages 1-6, Istanbul, Turkey, August 2015.

(P21) V. Estrada-Manzo, Zs. Lendek, T. M. Guerra, Improving the observer design for discrete-time TS descriptor models under the quadratic framework. In Proceedings of the IFAC Conference on Embedded Systems, Computational Intelligence and Telematics in Control, pages 276-281, Maribor, Slovenia, June 2015.

(P22) V. Estrada-Manzo, Zs. Lendek, T. M. Guerra, Output feedback control for T-S discrete-time nonlinear descriptor models. In Proceedings of the 53rd IEEE Conference on Decision and Control, pages 860-865, Los Angeles, CA, USA, December 2014.

(P23) V. Estrada-Manzo, Zs. Lendek, T. M. Guerra, H_∞ control for discrete-time Takagi-Sugeno descriptor models: a delayed approach. In Proceedings of the

23e Rencontres Francophones sur la Logique Floue et ses Applications, pages 1-6, Ajaccio, France, October 2014.

(P24) V. Estrada-Manzo, Zs. Lendek, T. M. Guerra, Discrete-time Takagi-Sugeno descriptor models: observer design. In Preprints of the 2014 IFAC World Congress, pages 7965-7969, Cape Town, South Africa, August 2014.

(P25) V. Estrada-Manzo, T. M. Guerra, Zs. Lendek, Discrete-time Takagi-Sugeno descriptor models: controller design. In Proceedings of the 2014 IEEE World Congress on Computational Intelligence, IEEE International Conference on Fuzzy Systems, pages 1-5, Beijing, China, July 2014.

(P26) V. Estrada-Manzo, T. M. Guerra, Zs. Lendek, An LMI approach for observer design for Takagi-Sugeno descriptor models. In Proceedings of the 2014 IEEE International Conference on Automation, Quality and Testing, Robotics, pages 1-6, Cluj-Napoca, Romania, May 2014.

(P27) V. Estrada-Manzo, T. M. Guerra, Zs. Lendek, M. Bernal, Improvements on non-quadratic stabilization of continuous-time Takagi-Sugeno descriptor models. In Proceedings of the 2013 IEEE International Conference on Fuzzy Systems, pages 1-6, Hyderabad, India, July 2013.

12.2 Applications

Together with students S. Beyhan, A. Berna, J. Guzman-Gimenez, P. Petrehus, Z. Nagy, E. Páll, and V. Estrada-Manzo, the developed methods have been tested in simulation and in real-world experiments for the control and observation of several processes, mostly robotic applications.

For instance, we have considered the control and estimation problems for a robotic manipulator that transports a varying payload. We have tested adaptive fuzzy and sliding-mode control in (P38), fuzzy observers and extended Kalman filters, respectively in (P35), adaptive control in (P32). For a 2DOF robot arm, we have developed an robust controller coupled with an adaptive observer in (P36), and, to estimate the torque in the absence of available measurements, an unknown input observer in (P30). The modeling and design of observer and controller for a 3D crane has been presented in (P34). Finally, control of a quadcopter has been presented in e.g., (P37) and (P29).

The above mentioned work was presented in the following publications:

(P28) V. Estrada-Manzo, Zs. Lendek, T. M. Guerra, Observer Design for Robotic Systems via Takagi-Sugeno Models and Linear Matrix Inequalities. In Handling uncertainty and networked structure in robot control, series Studies in Systems, Decision and Control, L. Busoniu and L. Tamas, Editors, pages 103-128. Springer International Publishing, 2015.

(P29) Zs. Lendek, A. Sala, P. Garcia, R. Sanchis, Experimental application of Takagi-Sugeno observers and controllers in a nonlinear electromechanical system. Journal of Control Engineering and Applied Informatics, vol. 15 , no. 4, pages 3-14, 2013.

(P30) Z. Nagy, E. Pall, Zs. Lendek, Unknown input observer for a robot arm using TS fuzzy descriptor models. In Proceedings of the 2017 IEEE Conference on Control Technology and Applications, pages 939-944, Hawaii, USA, August 2017.

(P31) Z. Nagy, Zs. Lendek, Quadcopter modeling and control. In Proceedings of the Journees Francophones sur la Planification, la Decision et l'Apprentissage pour la conduite de systemes, pages 1-2, Caen, France, July 2017

(P32) S. Beyhan, F. Sarabi, Zs. Lendek, R. Babuska, Takagi-Sugeno fuzzy payload estimation and adaptive control. In Preprints of the 2017 IFAC World Congress, pages 867-872, Toulouse, France, July 2017.

(P33) Z. Nagy, Zs. Lendek, Takagi Sugeno fuzzy modelling and control of a robot arm. In Proceedings of the 17th International Conference on Energetics-Electrical Engineering, 26th International Conference on Computers and Education, pages 265-270, Cluj-Napoca, Romania, October 2016.

(P34) P. Petrehus, Zs. Lendek, P. Raica, Fuzzy modeling and control of a 3D crane. In Proceedings of the 2013 IFAC International Conference on Intelligent Control and Automation Science, pages 201-206, Chengdu, China, September 2013.

(P35) S. Beyhan, Zs. Lendek, M. Alci, R. Babuska, Takagi-Sugeno Fuzzy Observer and Extended Kalman Filter for Adaptive Payload Estimation. In Asian Control Conference, pages 1-6, Istanbul, Turkey, June 2013.

(P36) S. Bindiganavile Nagesh, Zs. Lendek, A. A. Khalate, R. Babuska, Adaptive fuzzy observer and robust controller for a 2-DOF robot arm. In Proceedings of the 2012 IEEE World Congress on Computational Intelligence, IEEE International Conference on Fuzzy Systems, pages 149-155, Brisbane, Australia, June 2012.

(P37) Zs. Lendek, A. Berna, J. Guzman-Gimenez, A. Sala, P. Garcia., Application of Takagi-Sugeno observers for state estimation in a quadrotor. In Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference, pages 7530-7535, Orlando, Florida, December 2011.

(P38) S. Beyhan, Zs. Lendek, R. Babuska, M. Wisse, M. Alci, Control of a Robot Manipulator with Varying Payload Mass. In Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference, pages 8291-8296, Orlando, Florida, December 2011.

Next to mechanical systems, the developed – usually observer design – methods have been tested in diverse applications, such as for traffic models in (P41), in a hopper dredger in (P42) or a water treatment process in (P40). An ongoing research direction is the development of efficient model and controller for the air-fuel ratio (P39).

(P39) T. Laurain, Zs. Lendek, J. Lauber, R. M. Palhares, A new air-fuel ratio model fixing the transport delay: validation and control. In Proceedings of the 2017 IEEE Conference on Control Technology and Applications, pages 1904-1909, Hawaii, USA, August 2017.

(P40) N. Hodasz, V. Bradila, I. Nascu, Zs. Lendek, Modeling and parameter estimation for an activated sludge wastewater treatment process. In Proceedings of the 2016 IEEE International Conference on Automation, Quality and Testing, Robotics, pages 1-6, Cluj-Napoca, Romania, May 2016.

(P41) Z. Hidayat, Zs. Lendek, R. Babuska, B. De Schutter, Fuzzy observer for state estimation of the METANET traffic model. In Proceedings of the 13th International IEEE Conference on Intelligent Transportation Systems, pages 19-24, Madeira, Portugal, September 2010.

(P42) P. Stano, Zs. Lendek, R. Babuska, J. Braaksma and C. de Keizer, Particle Filters for the Estimation of the Average Grain Diameter of the Material Excavated by a Hopper Dredger. In Proceedings of the 2010 IEEE Multi-Conference on Systems and Control, pages 292-297, Yokohama, Japan, September 2010.

Chapter 13

Overall plans for the future

13.1 Introduction

My research work aims at developing control and estimation methods for complex systems. This is motivated by the high complexity of modern controlled systems, manifested in properties such as nonlinearity, time-variance, stochasticity and distributed or switching nature. Automated systems are increasingly expected to perform well in complex and unpredictable environments, such as in the everyday surroundings of human beings. Thus, advanced, nonlinear control algorithms that can handle such uncertainties are necessary, which in turn rely on more and more complex descriptions of the systems. For such systems, I focused on the promising linear parameter-varying or Takagi-Sugeno model representations, which in principle can adapt the advantages of linear time-invariant system-based design to nonlinear systems.

However, the control and estimation problems remain highly challenging due to the computational limitations and conservatism of existing methods. I have reduced this conservativeness by incorporating knowledge of the structure of the model in the design methods. My recent work focused strongly on design methods for systems exhibiting a structure, such as distributed, switching or with specific interconnection structure, with the goal of incorporating the known structure in the design steps and in this way reduce the conservativeness of the conditions for computing the controller. Furthermore, recently I have considered the development of local results – thus starting the transition to general nonlinear systems.

In my future work I will rely on my expertise in the analysis and development of both control and estimation methods for nonlinear systems, particularly in the Takagi-Sugeno framework. I plan to develop a comprehensive set of algorithmic tools for the automatic observer and controller design for a wide array of systems, accompanied by analytical performance guarantees, as well as by practical applications. These results will serve as a solid platform from which to explore new directions.

13.2 Research plan

As a general theme for the upcoming three to five years, I will start from the research presented in this thesis. The following major topics will be addressed:

- *Adaptive observers for control*

  I am planning to study adaptive observers both for parameter identification and unknown input estimation. This line of research would be based on the idea of building a model for estimation purposes, instead of in order to control it, as is done in indirect adaptive control. The observer is in effect designed based on the model built. This can also be seen as evolving fuzzy systems. However, one of the shortcoming of the evolving systems is that, due to the membership functions used they require pruning of the rules. To avoid this, I propose the parameterizations of the rules, in effect the membership functions, such that a given nonlinear system, defined on a compact set can be represented using a minimum number of rules.
- *Networked control*

  Networked systems are becoming extremely important in todays world: communication networks, power and transport grids, decentralized computing networks, and social networks are just some examples of such systems influencing everyday life. Such systems present several challenges, among which packet losses, delays, bandwidth limits, etc. In the last few years several results have been developed using a Takagi-Sugeno representation of networked systems, usually employing a Lyapunov-Krasovskii functional with multiple integrals. The conditions generally require an upper bound on the derivative of the membership functions, which will need to be verified a posteriori. Thus, I will investigate whether the conditions on the upper bound or similar ones can be included in the design, leading to a local result, but without the need of further verification.
- *Controllability and observability*

  Current approaches for both observer and controller design in fuzzy systems (or, more general in multiple-model approaches) rely on the observability/ controllability of the local models, not on the properties of the nonlinear system. Depending on how the local models are obtained, their properties may have nothing in common with those of the underlying nonlinear system. A topic of interest is to investigate under which conditions are the same properties obtained and/or develop a constructive method to obtain an exact multiple-model representation that also preserves the properties of the nonlinear system.
- *Applications*

  One of my goals is to apply the results of my research in practice. I have already been involved in several applicative projects and many of the topics are strongly motivated by practical applications. For instance, in machine tool and manipulator applications, the cutting force exerted by the tool or the exerting force/torque of the robot is of interest, but is very difficult or expensive to measure (Corless and Tu, 1998; Ha and Trinh, 2004). Load estimation in e.g., electricity distribution networks (Sheldrake, 2005) or wind turbines (Li and Chen, 2005) is necessary for proper planning and operation. In biomechanics, the myoskeletal system can be regarded as a dynamic system, where segment positions and trajectories are the system outputs and joint torques are the non-measurable inputs (Guelton et al., 2008). Examples of networked systems which require distributed estimation and control include sensor networks, multi-robot teams, adaptive optics, electrical power plants and petrochemical processes, traffic networks, energy and water infrastructures, etc.

I will validate the fundamental and algorithmic contributions described above in real-life case studies. Cooperation with industrial partners will be sought, with the longer-term goal of commercial deployment.

13.3 Long-term research goals

On the longer term, I wish to develop a unified framework by bridging the gap between fuzzy, linear parameter varying, piecewise linear, etc. systems and make the leap to general nonlinear systems. I will have built a strong research group that is able to develop mathematically rigorous research that has also practical attractiveness. I will continue attracting and recruiting top undergraduate students, and I will exploit public funding opportunities at the national, European, and international levels, as well as industrial funding with local and international companies. Although the main focus will be on fundamental research, applications to domains such as robotics, transportation, medical treatment, etc. will also be made clear, such that the research attracts funding from both industry and funding agencies.

Although the background in fuzzy systems provide a good starting point, I do not see my research interests limited to this area. In fact, with the knowledge that I developed in the last years I am already looking at new research opportunities in the broader systems and control field.

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