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HABILITATION THESIS

**Some contributions
to the study
of measure preserving maps
and approximation operators**

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Abstract

This habilitation thesis, entitled *Some contributions to the study of measure preserving maps and approximation operators*, contains some results obtained by the author in the field of Mathematical Analysis after obtaining his Ph.D. degree in Mathematics at Babeş-Bolyai University. On one hand, it continues the study of approximation of functions by positive linear operators done by the author in the period of his doctoral studies (2006-2010) under the supervision of Professor dr. Ioan Gavrea. On the other hand, the field of study is extended to the construction of maps which preserve the area or the volume. The results in this field are mainly obtained in collaboration with my colleague, Prof. Dr. Daniela Roşca.

The thesis has two distinct parts. The first part contains some of the scientific achievements obtained by the author in the last ten years. For the sake of brevity and unity, not all the mathematical results of the author are presented in this thesis, but only the most significant. The second part presents some future plans for the academic career.

We start in Chapter 1 with the measure preserving maps. Each section contains results from the six papers published in this field. We construct area-preserving maps (or volume-preserving maps) and obtain hierarchical grids on the sphere (or ball) which can be used in geophysics, astronomy, crystallography, medical imaging, computer graphics, statistical applications or other fields. A uniform grid on a two (three)-dimensional domain is a division of the surface (or volume) into cells that have the same measure. When a grid is not fine enough to solve a problem accurately, we need some refinement of the grid. We obtain uniform grids which can be refined by a simple procedure.

In the first section of Chapter 1, we present a bijective continuous area preserving map from a class of elongated dipyrramids to the sphere and its inverse. Using these maps one can obtain uniform and refinable grids on the sphere. As a particular case, we show that HEALPix grids (which are used in Astrophysics) can be obtained from these maps. These grids can also be used to obtain configurations of points on the sphere. For each configuration of points, some numbers (called „energies”) can be calculated that provide information about that configuration. An open problem in physics is how to arrange a given number of points on the sphere so that the energy of the configuration is minimal. Related to this

problem, Steve Smale proposed in 1998 a list of unsolved problems in mathematics, which includes that of finding an algorithm to generate, for a given number, one configuration of points on the sphere, that is near-optimal. We show that our configurations of points are near-optimal. The area preserving map can also be used to obtain a volume preserving map on the ball.

In Section 2 of Chapter 1, we present an area-preserving map from the regular octahedron to the sphere. The method used to obtain the map is the same, and the results are similar, but I have included these results in the present thesis for one particular property that the associated grids have and, in general, it is hard to prove: the diameter boundedness. This property assures that in the process of hierarchical division of the cells, they don't become long and thin. Besides the near-optimal configuration of points which can be obtained from our grids, we give as a possible application of our map a construction of geographical square maps of a hemisphere. As can be seen from the citations of this article [50] on Google Scholar, a color mapping technique that provides an intuitive visualization of surface normals was patented in USA in 2022, based on our mapping.

In Section 3 of Chapter 1, we construct a volume preserving map from a cube to a tetrahedron and generalize this from a cube to an arbitrary tetrahedron, and further on arbitrary convex polyhedron. We also present how to construct uniform refinable 3D grids of the octahedron.

In Section 4 of Chapter 1, we construct two area-preserving maps from the square to the 2D p -ball (the domain defined by $|x|^p + |y|^p \leq |a|^p$). A discussion of the distortion of the cells is also included. In Section 5, we construct volume preserving maps between 3D p -balls.

In Section 6 of Chapter 1, we present how to obtain a volume-preserving map from a regular 3D octahedron to the 3D ball. We show how to construct other uniform refinable grids of the regular octahedron, and we give some remarks on their implementation. Finally, we obtain a multiresolution analysis of $L^2(K)$ (where K is the octahedron) and piecewise constant orthonormal wavelet bases on $L^2(B)$ (where B is the 3D ball).

In Chapter 2, we mention some results related to approximation operators. We have selected four relevant papers and present them in four separate sections. In Section 1, we study the Szász-Mirakyan operators of max-product type. They are positive sublinear operators defined based on the classical operators of Szász-Mirakyan-Favard by replacing the sum operator with the max operator. They can be used for the approximation of bounded or unbounded functions. For some classes of functions, they provide a better order of approximation. By using a suitable modulus of continuity, our results extend in two ways the existing results. First, a larger class of bounded functions f can be uniformly approximated, namely those for which $f(x^2)$ is uniformly continuous, and second, we can approximate unbounded functions with exponential growth.

Section 2 of Chapter 2 contains results related to a particular type of positive

linear operators generated by a differential equation called exponential-type operators. For these operators, we can obtain the asymptotic behavior of the moments and some general Voronovskaya-type results. Due to space requirements, we have limited our presentation to some general results, but we must mention that other general results can be given, and many more particular cases.

Section 3 of Chapter 2 presents the operators obtained from the classical Bernstein operators by squaring their basis and using normalization. Their study is recent in the mathematical literature. These kinds of operators obtained from classical exponential-type operators by using a power of the basis are also of exponential type. So, some general results we have presented in Section 2 can be applied to these operators, too. In this way, we obtain new results which are not yet published. We have even answered a conjecture (see Remark 2.30).

In Section 4 of Chapter 2, we study all the exponential-type operators that can be generated by a power function. Among them, we reobtain the classical operators of Bernstein, of Szász-Favard-Mirakyan, of Baskakov, of Post-Widder, and of Gauss-Weierstrass. Using the Laplace and Fourier transforms, we obtain new operators that can be used for uniform approximation of functions.

The second part of the thesis contains Chapter 3, which gives some of the future plans of the author. I intend to continue my didactic and research activities at the Technical University of Cluj-Napoca. As regards teaching the undergraduate students, especially in this new context of AI, I want to improve the methods used in education, to involve students in an active and participatory way. I also plan to elaborate on research proposals and supervise master's and doctoral students.

I have had a good collaboration with my colleague Daniela Roșca, and this thesis presents five papers written together. I have also written a paper in collaboration with a team from the electrical engineering department and a joint paper with Vijay Gupta. Another recent paper has been written in collaboration with a research team from Harran University in Turkey. These prove that the author can successfully work in teams, on the national and international level. I intend to continue these collaborations in the future and to start new ones.

I plan to publish a monograph, which contains all the results obtained so far and new results that are expected to appear in future projects. Some of these future projects I will describe briefly in the following.

I want to extend the study of exponential-type operators in several directions. The generating differential equation of these operators can be given in a more general form to include other known types of positive linear operators. Not all of the positive linear operators are of exponential type. Is it possible to extend the differential equation to a certain relation that will allow the inclusion in the general class of operators of other modifications of the classical operators (like that of Chlodovsky, Kantorovich, or Durrmeyer)? What is the most general form of this equation that can be considered is a question for future research.

We have given some pointwise and asymptotic approximation results related to the exponential-type operators. What other properties can be derived directly from the differential equation that they verify? Can we obtain uniform approximation results for this general class of operators? What can we say about the maximal class of functions that are approximated by a given sequence of exponential-type operators? How can we determine it from the equation?

One class of exponential-type operators that we began to study is that of operators constructed by using the square of the basis of some classical operators. What functions can be chosen instead of the power function such that the operators we obtain are of exponential type? What are the properties of these new operators? Do they have good approximation properties? This is another direction of future research.

In the field of measure-preserving maps, we want to construct, if it is possible, a symmetric map in all the variables, one that shows a symmetry with respect to the domain. Starting from this, we want to construct symmetric and easy-to-implement wavelets.