Domain: MATHEMATICS

HABILITATION THESIS
– ABSTRACT –

COHOMOLOGICAL METHODS IN MODULAR REPRESENTATION THEORY AND RELATED TOPICS

Assoc. Prof. Constantin-Cosmin TODEA PhD
Faculty of AUTOMATION and COMPUTERS SCIENCE
TECHNICAL University of CLUJ-NAPOCA

– Cluj-Napoca –
2024

This thesis is about the results, obtained after defending my PhD in 2010, concerning modular representation theory of finite groups and methods in cohomology theory. Roughly speaking modular representation theory is the part of representation theory that studies linear representations of finite groups over a field k of positive prime characteristic p. When writing the papers that are summed up in this thesis, I was influenced by the title of a paper of Alperin: "Cohomology is representation theory". The main concepts that appear in this thesis are: the group algebra kG (of a finite group G over an algebraically closed field k of prime characteristic p), the block algebras in kG (a block is a primitive idempotent in the center $Z(kG)$ of the group algebra), the saturated fusion system (some category mimicking the behavior of fusions in finite groups), the cohomology of finite groups, the cohomology of saturated fusion systems and the Hochschild cohomology of algebras (the related cohomological invariants of the previous concepts). Broadly speaking cohomology theory is the study of the homological functor *Ext* and the related algebraic structures that it entail.

The thesis has six chapters. The First Chapter of this thesis is dedicated to providing a dictionary of the above concepts, notations and related ideas from modular representation theory and cohomology theory, that are used in the following chapters.

Chapter Two is about some results that are inspired by, and continue the work done for my PhD thesis. We deal with: cohomology rings of saturated fusion systems, restriction maps and varieties (Section 2.1), the image of some map in the cohomology of blocks of finite groups (Sasaki's Conjecture in Section 2.3) and first Hochschild cohomology of block algebras of finite groups (Section 2.4). From this chapter I would like to emphasize Theorem 2.2.3, which is Mislin's Theorem for cohomology of fusion systems. In a paper published in „Inventiones Mathematicae" (in 2017) Benson, Henke and Grodal were able to obtain an algebraic proof of a celebrated Theorem of Mislin: if an inclusion of finite groups H in G of index prime to p induces a homeomorphism of mod p cohomology varieties, then H controls p-fusion in G, if p is an odd prime. In Section 2.2 we give an algebraic proof for an extension of the above theorem to the case of cohomology of saturated fusion systems defined on p-groups again, if p is an odd prime.

In Chapter Three we are dealing with results that extend well-known concepts and theorems from: blocks of finite groups, G-interior algebras and basic Morita equivalences, to: group-invariant blocks of normal subgroups, H-interior G-algebras (H is a normal subgroup of a finite group G) and group graded basic Morita equivalences, respectively. Section 3.1 is relying on a joint work with Coconeț and, is about the construction of a generalization of the extended Brauer quotient defined by Puig and Zhou in 2007 to the case of H-interior G-algebras. We use this extended Brauer quotient on p-permutation algebras to establish some correspondences (Theorem 3.1.7) that generalize known results holding in the group algebra case. Section 3.2 is about a generalization of blocks, that is the primitive idempotents of *(kH)^G*, where *(kH)^G* is the subalgebra of G-stable elements in kH with G acting by conjugation on it. We are concerned with fusion systems associated in this context and the invariance of some fusion systems Theorem 3.2.7, if the associated hyperfocal subgroup is abelian. The second main result of this section (Theorem 3.2.8) is related to group graded basic Morita equivalences, which were recalled in Chapter One, Section 1.1.2. In Section 3.4, that is based again on some joint work with Coconeț, we return to the classical case of blocks (hence H=G in the setup of Section 3.2) in order to investigate their relation with respect to blocks of factor groups (called dominating blocks).

Chapter Four is especially about homological algebra: group cohomology and Hochschild cohomology in various contexts. However in some sections we refer and we are foreseeing applications to block algebras in the context of modular representation theory.

Symmetric cohomology of groups, which is the main object of study in Section 4.1, was introduced by M. Staic in 2009 in order to associate to topological spaces some elements in the third symmetric cohomology of some groups. In Section 4.1 we show that there is a well-defined restriction, conjugation and transfer map in symmetric cohomology of groups (Definition 4.1.2), which form a Mackey functor (Theorem 4.1.3), if we impose an extra condition (the injectivity of the natural map from symmetric group cohomology to group cohomology). In Section 4.2 we define the symmetric Hochschild cohomology of twisted group algebras (Definition 4.2.1) similarly to Staic's construction of symmetric group cohomology. This was realized by the construction of an action of the symmetric group on the Hochschild cochain complex of a twisted group algebra with coefficients in a bimodule. In the main results (Theorems 4.2.5 and 4.2.7), that have been published in 2022 in paper published in „Homology, Homotopy and Applications", we gave explicit embeddings and connecting homomorphisms between the symmetric group cohomology and the symmetric Hochschild cohomology of twisted group algebras. Section 4.3 is of a different flavor with respect to Sections 4.1 and 4.2. This section is based on the recent article published as unique author in Comptes Rendu Mathematique, in 2021. The Hochschild cohomology group of any associative algebra has a rich structure. It is a graded commutative algebra via the cup product and, it has a graded Lie bracket of degree *-1* obtaining what is now called Gerstenhaber algebra. A new structure in Hochschild theory, BV algebra, has been extensively studied in topology and mathematical physics for a long time. Roughly speaking a Batalin-Vilkovisky structure is an operator on Hochschild cohomology which squares to zero and which, together with the cup product, can express the Gerstenhaber bracket. Other structures called BD algebras (Beilinson-Drinfeld algebras), that appear in mathematical physics, have a superficial similarity with BV algebras. The main goal of this section (Theorem 4.3.6) is to introduce BD algebra structure from mathematical physics into algebra, by obtaining methods of constructing BD algebras in the context of group cohomology. An important ingredient for achieving this is Theorem 4.3.4, where we generalize the well-known fact that the group cohomology ring $H^*(G, \mathbf{Z}/p\mathbf{Z})$ together with a connecting homomorphism (called Bockstein map), that is induced by a given short exact sequence of trivial $\mathbf{Z}G$-modules, is a DG-algebra. We are also able to provide in Section 4.3.3 an example for which Theorem 4.3.6 applies.

The results presented in Chapter Five are related to some aspects of concepts and ideas that appear in the above chapters, however they are treated from a different perspective. This chapter is based on an article published in „Journal of Algebra" in 2012 and on the joint work with Coconeț and Marcus, published in 2017. In Section 5.1 we use the theory of inverse semigroups to identify and define a class of symmetric algebras (Theorem 5.1.1), which we call inverse-symmetric algebras in Definition 5.1.1 Recall that a k-algebra A (in this section we assume that k is a commutative ring) is called symmetric if there is a k-linear central form from A to k inducing an isomorphism of (A, A)-bimodules from A to the k-dual $Hom(A,k)$. Section 5.2 is about induction of algebras. In finite group representation theory a notion of Frobenius induction for interior G-algebras (here G is a finite group) was introduced by Puig. This construction has many important uses, and it is strongly related to the classical Frobenius induction of modules. Let k be a field , let A, B be two k-algebras and let C be a B-interior k-algebra. If M is an (A,B)-bimodule, Linckelmann defined in 2002 the induced algebra $Ind\_M(C)$. The first goal of this section is to give conditions on the (A,B)-bimodule M such that the induced algebra $Ind\_M(C)$ can be expressed in two ways: as an endomorphism algebra and as a tensor product (Theorem 5.2.1). The second aim is to define a non-injective induction through a homomorphism of augmented algebras (Definition 5.2.1) that generalizes Puig's non-injective induction and, to generalize Turull's induction of a H-

algebra from a subgroup H to a finite group G, to the context of Hopf module algebras. The third objective is to find a relationship through the smash product (in the context of Hopf algebras) between the above two types of induction (Theorems 5.2.4 and 5.2.6).

The final part of the thesis, Chapter Six, is devoted to plans for evolutions of research and academic directions, as required by the Technical University of Cluj-Napoca habilitation format. I was involved, as a member, in a consistent number of UEFISCDI research grants but also, I was the director of a „Young Teams" research UEFISCDI grant, from 2019 until 2021 (collaboration with UBB). The Bibliography Section contains 100 references. Through hese references we can find 17 ISI-Web of Science indexed, published articles such that: the author of this thesis is the unique author of 10 of these articles and he is a coauthor in 7 of these references.